

Research Article

# On Fuzzy Baire-Separated Spaces and Related Concepts

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## Abstract

In this paper, a new class of fuzzy topological spaces, namely fuzzy Baire-separated spaces is introduced in terms of fuzzy Baire sets. Several characterizations of fuzzy Baire-separated spaces are established. It is shown that fuzzy Baire sets lie between disjoint fuzzy P-sets and fuzzy  $F_\sigma$ -sets in a fuzzy Baire-separated space. Conditions under which fuzzy topological spaces become fuzzy Baire-separated spaces are established. Fuzzy nowhere dense sets are fuzzy closed sets in fuzzy nodec spaces and subsequently a question will arise. Which fuzzy topological spaces [other than fuzzy hyperconnected spaces, fuzzy globally disconnected spaces] have fuzzy closed sets with fuzzy nowhere denseness? For this, fuzzy topological spaces having fuzzy closed sets with fuzzy nowhere denseness are identified in this paper. It is verified that fuzzy ultraconnected spaces are non fuzzy Baire -separated spaces. The means, by which fuzzy weakly Baire space become fuzzy Baire -separated spaces and in turn fuzzy Baire - separated spaces become fuzzy seminormal spaces, are obtained. There are scope in this paper for exploring the inter-relations between fuzzy Baire spaces and Baire -separated spaces.

## Keywords

Fuzzy Nowhere Dense Set, Fuzzy Residual Set, Fuzzy Baire Set, Fuzzy Weakly Baire Space, Fuzzy Strongly Baire Space, Fuzzy Nodec Space, Fuzzy Seminormal Space, Fuzzy Globally Disconnected Space

## 1. Introduction

Any application of mathematical notions depends firmly how one introduces basic ideas that may yield various theories in various directions. If the basic idea is appropriately introduced, then not only the existing theories stand but also the possibility of emerging new theories increases. On these lines, the notion of fuzzy sets as a new approach for modelling uncertainties was introduced by L. A. Zadeh [26] in 1965. The concept of fuzzy topological space was introduced by C. L. Chang [5] in 1968. Based on this concept, many studies have been conducted in general theoretical areas and in various application sides.

The notion of Baire sets in classical topology was introduced and studied by Andrzej Szymanski [2]. The concept of fuzzy Baire sets in fuzzy topological spaces was introduced and studied by G. Thangaraj and R. Palani [14] in terms of fuzzy open sets and fuzzy residual sets and further studied in [18]. In classical topology, W. Adamski [1] introduced the concept of Baire-separation in topological spaces. Motivated on these lines, the notion of fuzzy Baire - separation in fuzzy topological spaces is introduced in terms of fuzzy Baire sets. Conditions under which fuzzy topological spaces become fuzzy Baire-separated spaces, are obtained. The fuzzy topo-

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logical spaces having fuzzy closed sets with fuzzy nowhere denseness are identified. It is found that fuzzy ultraconnected spaces are non fuzzy Baire -separated spaces. Condition under which a fuzzy weakly Baire space becomes a fuzzy Baire-separated space is obtained. Also the means by which a fuzzy Baire -separated space becomes a fuzzy semi normal space is also obtained.

## 2. Preliminaries

Several basic notions and results used in the sequel, are given for making the exposition self - contained. In this work by  $(X, T)$  or simply by  $X$ , we will denote a fuzzy topological space due to C. L. Chang (1968). Let  $X$  be a non-empty set and  $I$ , the unit interval  $[0, 1]$ . A fuzzy set  $\lambda$  in  $X$  is a function from  $X$  into  $I$ . The fuzzy set  $0_X$  is defined as  $0_X(x) = 0$  for all  $x \in X$  and the fuzzy set  $1_X$  is defined as  $1_X(x) = 1$ , for all  $x \in X$ .

Definition 2.1 [5]: A fuzzy topology is a family  $T$  of fuzzy sets in  $X$  which satisfies the following conditions:

- (a).  $0_X \in T$  and  $1_X \in T$ ,
- (b). If  $A, B \in T$ , then  $A \wedge B \in T$ ,
- (c). If  $A_i \in T$  for each  $i \in J$ , then  $\bigvee_i A_i \in T$ .

$T$  is called a fuzzy topology for  $X$  and the pair  $(X, T)$  is a fuzzy topological space, or  $fts$  for short. Members of  $T$  are called fuzzy open sets of  $X$  and their complements, are called fuzzy closed sets in  $X$ .

Definition 2.2 [5]: Let  $(X, T)$  be a fuzzy topological space and  $\lambda$  be any fuzzy set defined on  $X$ . The fuzzy interior, the fuzzy closure and the complement of  $\lambda$  are defined respectively as follows:

- (i).  $\text{int}(\lambda) = \bigvee \{ \mu / \mu \leq \lambda, \mu \in T \}$ ;
- (ii).  $\text{cl}(\lambda) = \bigwedge \{ \mu / \lambda \leq \mu, 1 - \mu \in T \}$ ;
- (iii).  $\lambda'(x) = 1 - \lambda(x)$ , for all  $x \in X$ .

For a family  $\{\lambda_i / i \in I\}$  of fuzzy sets in  $(X, T)$ , the union  $\psi = \bigvee_i (\lambda_i)$  and the intersection  $\delta = \bigwedge_i (\lambda_i)$ , are defined respectively as

- (iv).  $\psi(x) = \sup_{i \in I} \{ \lambda_i(x) / x \in X \}$ ;
- (v).  $\delta(x) = \inf_{i \in I} \{ \lambda_i(x) / x \in X \}$ .

Lemma 2.1 [3]: For a fuzzy set  $\lambda$  of a fuzzy topological space  $X$ ,

- (i).  $1 - \text{int}(\lambda) = \text{cl}(1 - \lambda)$  and (ii).  $1 - \text{cl}(\lambda) = \text{int}(1 - \lambda)$ .

Definition 2.3: A fuzzy set  $\lambda$  in a fuzzy topological space  $(X, T)$  is called a

- (i). fuzzy  $G_\delta$ -set in  $(X, T)$  if  $\lambda = \bigwedge_{i=1}^\infty (\lambda_i)$ , where  $\lambda_i \in T$ ; fuzzy  $F_\sigma$ -set in  $(X, T)$  if  $\lambda = \bigvee_{i=1}^\infty (\lambda_i)$ , where  $1 - \lambda_i \in T$  [4].
- (ii). fuzzy regular-open set in  $(X, T)$  if  $\lambda = \text{int} \text{cl}(\lambda)$ ; fuzzy regular-closed set in  $(X, T)$  if  $\lambda = \text{cl} \text{int}(\lambda)$  [3].
- (iii). fuzzy semi - open set in  $(X, T)$  if  $\lambda \leq \text{cl} \text{int}(\lambda)$ ; fuzzy semi - closed set in  $(X, T)$  if  $\text{int} \text{cl}(\lambda) \leq \lambda$  [3].
- (iv). fuzzy dense set in  $(X, T)$  if there exists no fuzzy closed set  $\mu$  in  $(X, T)$  such that  $\lambda < \mu < 1$ . That is,  $\text{cl}(\lambda) = 1$ , in  $(X, T)$  [9].
- (v). fuzzy nowhere dense set in  $(X, T)$  if there exists no non-zero fuzzy open set  $\mu$  in  $(X, T)$  such that  $\mu < \text{cl}(\lambda)$ . That is,

$\text{int} \text{cl}(\lambda) = 0$ , in  $(X, T)$  [9].

(vi). fuzzy first category set in  $(X, T)$  if  $\lambda = \bigvee_{i=1}^\infty (\lambda_i)$ , where  $(\lambda_i)$ 's are fuzzy nowhere dense sets in  $(X, T)$ . Any other fuzzy set in  $(X, T)$  is said to be of fuzzy second category [9].

(vii). fuzzy residual set in  $(X, T)$  if  $1 - \lambda$  is a fuzzy first category set in  $(X, T)$  [10].

(viii). fuzzy Baire set in  $(X, T)$  if  $\lambda = \mu \wedge \delta$ , where  $\mu$  is a fuzzy open set and  $\delta$  is a fuzzy residual set in  $(X, T)$  [14].

(ix). fuzzy  $\sigma$ -nowhere dense set in  $(X, T)$  if  $\lambda$  is a fuzzy  $F_\sigma$ -set in  $(X, T)$  such that  $\text{int}(\lambda) = 0$  [12].

(x). fuzzy  $\sigma$ -boundary set in  $(X, T)$  if  $\lambda = \bigvee_{i=1}^\infty (\lambda_i)$ , where  $\mu_i = \text{cl}(\lambda_i) \wedge (1 - \lambda_i)$  and  $(\lambda_i)$ 's are fuzzy regular open sets in  $(X, T)$  [15].

(xi). fuzzy strongly nowhere dense set in  $(X, T)$  if  $\lambda \wedge (1 - \lambda)$  is a fuzzy nowhere dense set in  $(X, T)$  [16].

Definition 2.4 [22]: Let  $(X, T)$  be a fuzzy topological space. A fuzzy closed set  $\lambda$  in  $X$  is called a fuzzy  $P$ -set if  $\lambda \leq 1 - \mu$ , where  $\mu$  is a fuzzy  $F_\sigma$ -set in  $(X, T)$  implies that  $\lambda \leq 1 - \text{cl}(\mu)$  in  $X$ .

Definition 2.5: A fuzzy topological space  $(X, T)$  is called a

- (i). fuzzy submaximal space if for each fuzzy set  $\lambda$  in  $X$ , such that  $\text{cl}(\lambda) = 1$ ,  $\lambda \in T$  [4].
- (ii). fuzzy hyperconnected space if every non- null fuzzy open subset of  $(X, T)$  is fuzzy dense in  $(X, T)$  [6].
- (iii). fuzzy extraresolvable space if for  $\lambda_i$  and  $\lambda_j (i \neq j)$  are fuzzy dense sets in  $X$ , then  $\lambda_i \wedge \lambda_j$  is a fuzzy nowhere dense set in  $X$  [25].
- (iv). fuzzy nodec space if each fuzzy nowhere dense set is a fuzzy closed set in  $X$  [11].
- (v). fuzzy ultraconnected space if whenever  $\lambda$  and  $\mu$  are two non-zero fuzzy closed sets in  $(X, T)$ ,  $\lambda \not\leq 1 - \mu$ , in  $(X, T)$  [23].
- (vi). fuzzy globally disconnected space if each fuzzy semi-open set is fuzzy open in  $X$  [20].
- (vii). fuzzy Baire space if  $\text{int}(\bigvee_{i=1}^\infty (\lambda_i)) = 0$ , where  $(\lambda_i)$ 's are fuzzy nowhere dense sets in  $X$  [10].
- (viii). fuzzy weakly Baire space if  $\text{int}(\bigvee_{i=1}^\infty (\mu_i)) = 0$ , where  $\mu_i = \text{cl}(\lambda_i) \wedge (1 - \lambda_i)$  and  $(\lambda_i)$ 's are fuzzy regular open sets in  $X$  [15].

(ix). fuzzy strongly Baire space if  $\text{cl}(\bigvee_{i=1}^\infty (\lambda_i)) = 1$ , where  $(\lambda_i)$ 's are fuzzy strongly nowhere dense sets in  $X$  [16].

(x). fuzzy perfectly disconnected space if for any two non-zero fuzzy sets  $\lambda$  and  $\mu$  defined on  $X$  with  $\lambda \leq 1 - \mu$ ,  $\text{cl}(\lambda) \leq 1 - \text{cl}(\mu)$  in  $X$  [19].

(xi). fuzzy  $F'$ -space if  $\lambda \leq 1 - \mu$ , where  $\lambda$  and  $\mu$  are fuzzy  $F_\sigma$ -sets in  $(X, T)$ , then  $\text{cl}(\lambda) \leq 1 - \text{cl}(\mu)$ , in  $X$  [21].

(xii). weak fuzzy  $Oz$ -space if for each fuzzy  $F_\sigma$ -set  $\delta$  in  $(X, T)$ ,  $\text{cl}(\delta)$  is a fuzzy  $G_\delta$ -set in  $(X, T)$  [24].

(xiii). fuzzy semi normal space if given a fuzzy closed set  $\lambda$  and a fuzzy open set  $\mu$  such that  $\lambda \leq \mu$ , then there exists a fuzzy regular open set  $\sigma$  such that  $\lambda \leq \sigma \leq \mu$  [7].

Theorem 2.1 [17]: If  $\lambda$  is a fuzzy Baire set in a fuzzy globally disconnected space  $(X, T)$ , then  $\lambda$  is a fuzzy  $G_\delta$ -set in  $(X, T)$ .

Theorem 2.2 [17]: If  $\lambda$  is a fuzzy Baire set in a fuzzy sub-maximal space  $(X, T)$ , then  $\lambda$  is a fuzzy  $G_\delta$ -set in  $(X, T)$ .

Theorem 2.3 [25]: If  $\lambda$  is a fuzzy residual set in a fuzzy extraresolvable space  $(X, T)$ , then  $\text{int}(\lambda) = 0$ , in  $X$ .

Theorem 2.4 [13]: If  $\lambda$  is a fuzzy residual set in a fuzzy topological space  $(X, T)$ , then there exists a fuzzy  $G_\delta$ -set  $\mu$  in  $(X, T)$  such that  $\mu \leq \lambda$ .

Theorem 2.5 [23]: If  $\lambda$  and  $\mu$  are fuzzy closed sets in a fuzzy hyperconnected space  $(X, T)$ , then  $\lambda$  and  $\mu$  are fuzzy nowhere dense sets in  $(X, T)$ .

Theorem 2.6 [22]: If a fuzzy set  $\lambda$  is a fuzzy P-set in a fuzzy topological space  $(X, T)$  such that  $\lambda \leq 1 - \mu$ , where  $\mu$  is a fuzzy  $F_\sigma$ -set in  $(X, T)$ , then there exists a fuzzy open set  $\delta$  in  $X$  such that  $\lambda \leq \delta \leq 1 - \text{int}(\mu)$ .

Theorem 2.7 [14]: If  $\lambda = \mu \wedge (1 - \alpha)$ , where  $\mu \in T$  and  $\alpha$  is a fuzzy  $\sigma$ -nowhere dense set in a fuzzy topological space  $(X, T)$ , then  $\lambda$  is a fuzzy Baire set in  $(X, T)$ .

Theorem 2.8 [15]: If  $\lambda$  is a fuzzy  $\sigma$ -boundary set in a fuzzy weakly Baire space  $(X, T)$ , then  $\lambda$  is a fuzzy  $\sigma$ -nowhere dense set in  $(X, T)$ .

Theorem 2.9 [16]: If  $\lambda$  is a fuzzy residual set in a fuzzy strongly Baire space, then  $\text{int}(\lambda) = 0$ , in  $X$ .

Theorem 2.10 [19]: If a fts  $(X, T)$  is a fuzzy perfectly disconnected space and  $\lambda \leq 1 - \mu$ , for any two non-zero fuzzy sets  $\lambda$  and  $\mu$  defined on  $X$ , then  $\text{cl}(\lambda) \neq 1$  and  $\text{cl}(\mu) \neq 1$ , in  $(X, T)$ .

Theorem 2.11 [10]: If  $\lambda$  is a fuzzy first category set in a fuzzy topological space  $(X, T)$ , then there is a fuzzy  $F_\sigma$ -set  $\eta$  in  $(X, T)$  such that  $\lambda \leq \eta$ .

Theorem 2.12 [8]: If a fuzzy topological space  $(X, T)$  is a fuzzy second category (but not fuzzy Baire) and fuzzy hyperconnected space and  $\lambda$  is a fuzzy residual set in  $(X, T)$ , then

- (i).  $\lambda$  is a fuzzy nowhere dense set in  $(X, T)$ ;
- (ii).  $(1 - \lambda)$  is a fuzzy dense set in  $(X, T)$ .

### 3. Baire – Separated Spaces

Definition 3.1: A fuzzy topological space  $(X, T)$  is called a fuzzy Baire-separated space if for each pair of fuzzy closed sets  $\mu_1$  and  $\mu_2$  in  $(X, T)$  such that  $\mu_1 \leq 1 - \mu_2$ , there exists a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $\mu_1 \leq \eta \leq 1 - \mu_2$ .

Example 3.1: Let  $X = \{a, b, c\}$ . Let  $I = [0, 1]$  and  $\alpha, \beta$  and  $\gamma$  are the fuzzy sets defined on  $X$  as follows:

$\alpha: X \rightarrow I$  is defined by  $\alpha(a) = 0.8; \alpha(b) = 0.7; \alpha(c) = 0.9$ ,

$\beta: X \rightarrow I$  is defined by  $\beta(a) = 0.6; \beta(b) = 0.9; \beta(c) = 0.8$ ,

$\gamma: X \rightarrow I$  is defined by  $\gamma(a) = 0.7; \gamma(b) = 0.6; \gamma(c) = 0.7$ .

Then,  $T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \beta \vee \gamma, \alpha \wedge \beta, \beta \wedge \gamma, \gamma \vee [\alpha \wedge \beta], 1\}$  is a fuzzy topology on  $X$ . By computation, one can find that  $\text{int}(1 - \alpha) = 0; \text{int}(1 - \beta) = 0; \text{int}(1 - \gamma) = 0;$

$\text{int}(1 - [\alpha \vee \beta]) = 0; \text{int}(1 - [\beta \vee \gamma]) = 0; \text{int}(1 - [\alpha \wedge \beta]) = 0; \text{int}(1 - [\beta \wedge \gamma]) = 0$  and

$\text{int}(1 - [\gamma \vee (\alpha \wedge \beta)]) = 0$ . Thus  $1 - \alpha, 1 - \beta, 1 - \gamma, 1 - (\alpha \vee \beta), 1 - (\beta \vee \gamma), 1 - (\alpha \wedge \beta), 1 - (\beta \wedge \gamma)$  and

$1 - (\gamma \vee [\alpha \wedge \beta])$  are fuzzy nowhere dense sets in  $(X, T)$ . By computation, one can find that  $1 - (\beta \wedge \gamma) = (1 - \alpha) \vee (1 - \beta) \vee (1 - \gamma) \vee (1 - (\alpha \vee \beta)) \vee (1 - (\beta \vee \gamma)) \vee (1 - (\alpha \wedge \beta))$  and thus  $1 - (\beta \wedge \gamma)$  is a fuzzy first category set in  $(X, T)$ . Then,  $\beta \wedge \gamma$  is a fuzzy residual set in  $(X, T)$ . For each fuzzy open set  $\delta = \alpha, \beta, \gamma, \alpha \vee \beta, \beta \vee \gamma, \alpha \wedge \beta, \beta \wedge \gamma, \gamma \vee [\alpha \wedge \beta]$ ,  $\delta \wedge [\beta \wedge \gamma] = \beta \wedge \gamma$  is a fuzzy Baire set in  $(X, T)$ . One can easily verify that for each pair of fuzzy closed sets  $\mu_1$  and  $\mu_2 \{1 - \alpha, 1 - \beta, 1 - \gamma, 1 - (\alpha \vee \beta), 1 - (\beta \vee \gamma), 1 - (\alpha \wedge \beta), 1 - (\beta \wedge \gamma), 1 - (\gamma \vee [\alpha \wedge \beta])\}$  in  $(X, T)$  such that  $\mu_1 \leq 1 - \mu_2$ , there exists a fuzzy Baire set  $\beta \wedge \gamma$  in  $(X, T)$  such that  $\mu_1 \leq \beta \wedge \gamma \leq 1 - \mu_2$ . Hence  $(X, T)$  is a fuzzy Baire-separated space.

Proposition 3.1: If  $\lambda$  is a fuzzy closed set and  $\delta$  is a fuzzy open set in a fuzzy Baire-separated space  $(X, T)$  such that  $\lambda \leq \delta$ , then there exists a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $\lambda \leq \eta \leq \delta$ .

Proof: Let  $\lambda$  be a fuzzy closed set and  $\delta$  be a fuzzy open set in  $X$ , such that  $\lambda \leq \delta$ . Then,  $\lambda \leq 1 - (1 - \delta)$ , where  $\lambda$  and  $(1 - \delta)$  are fuzzy closed sets in  $(X, T)$ . Since  $(X, T)$  is a fuzzy Baire-separated space, there exists a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $\lambda \leq \eta \leq 1 - (1 - \delta)$  and it follows that  $\lambda \leq \eta \leq \delta$ , in  $(X, T)$ .

Corollary 3.1: If  $\lambda$  is a fuzzy closed set and  $\delta$  is a fuzzy open set in a fuzzy Baire-separated space  $(X, T)$  such that  $\lambda \leq \delta$ , then there exists a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $\lambda \leq \eta \leq \delta$  and  $\eta$  is not a fuzzy dense set in  $X$ .

Proof: By proposition 3.1, there exists a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $\lambda \leq \eta \leq \delta$ , in  $(X, T)$ . Now,  $\text{cl}(\lambda) \leq \text{cl}(\eta) \leq \text{cl}(\delta)$  and then,  $\text{cl}(\eta) \leq \text{cl}(\delta)$  and by Lemma 2.1,  $\text{cl}(\delta)$  is a fuzzy regular closed set and thus  $\text{cl}(\delta)$  is a fuzzy closed set in  $(X, T)$ . Thus, there is a fuzzy closed set  $\text{cl}(\delta)$  in  $(X, T)$  such that  $\text{cl}(\eta) \leq \text{cl}(\delta) < 1$ . This implies that  $\text{cl}(\eta)$  is not a fuzzy dense set in  $(X, T)$ . Now  $\text{cl}(\text{cl}(\eta)) = \text{cl}(\eta)$  and  $\text{cl}(\text{cl}(\eta)) \neq 1$ , implies that  $\text{cl}(\eta) \neq 1$  and thus  $\eta$  is not a fuzzy dense set in  $X$ .

Proposition 3.2: If  $\lambda \vee \delta = 1$ , where  $\lambda$  and  $\delta$  are fuzzy open sets in a fuzzy Baire-separated space  $(X, T)$ , then there exists a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $1 - \lambda \leq \eta \leq \delta$ .

Proof: Suppose that  $\lambda \vee \delta = 1$ , where  $\lambda$  and  $\delta$  are fuzzy open sets in  $(X, T)$ . Then,  $1 - (\lambda \vee \delta) = 0$ . This implies that  $(1 - \lambda) \wedge (1 - \delta) = 0$  and then  $(1 - \lambda) \leq 1 - (1 - \delta)$ , where  $(1 - \lambda)$  and  $(1 - \delta)$  are fuzzy closed sets in  $(X, T)$ . Since  $(X, T)$  is a fuzzy Baire-separated space, there exists a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $1 - \lambda \leq \eta \leq 1 - (1 - \delta)$  and it follows that  $1 - \lambda \leq \eta \leq \delta$ , in  $(X, T)$ .

Proposition 3.3: If there exists a fuzzy Baire set  $\eta$  in a fuzzy topological space  $(X, T)$  such that  $\delta_1 \leq \eta \leq 1 - \delta_2$ , for each pair of disjoint fuzzy  $F_\sigma$ -sets  $\delta_1$  and  $\delta_2$ , then  $(X, T)$  is a fuzzy Baire-separated space.

Proof: Let  $\delta_1$  and  $\delta_2$  be a pair of disjoint fuzzy  $F_\sigma$ -sets in  $(X, T)$ . Then, it follows that  $\delta_1 \leq 1 - \delta_2$ . Since  $\delta_1$  and  $\delta_2$  are fuzzy  $F_\sigma$ -sets in  $(X, T)$ ,  $\delta_1 = \bigvee_{i=1}^{\infty} (\alpha_i)$  and

$\delta_2 = \bigvee_{k=1}^{\infty} (\beta_k)$ , where  $(\alpha_i)$ 's and  $(\beta_k)$ 's are fuzzy closed sets in  $(X, T)$ . Now  $\alpha_i \leq \delta_1$  and  $\beta_k \leq \delta_2$  and  $\alpha_i \leq \delta_1 \leq 1 - \delta_2 \leq 1 - \beta_k$ . By hypothesis, there exists a fuzzy Baire set  $\eta$  in  $X$  such that  $\delta_1 \leq \eta \leq 1 - \delta_2$ . Thus, for a pair of fuzzy closed sets  $\alpha_i$  and  $\beta_k$  in  $(X, T)$  such that  $\alpha_i \leq 1 - \beta_k$ , there exists a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $\alpha_i \leq \eta \leq 1 - \beta_k$  and hence  $(X, T)$  is a fuzzy Baire - separated space.

**Proposition 3.4:** If  $\lambda$  is a fuzzy closed set and  $\delta$  is a fuzzy open set in a fuzzy Baire - separated space  $(X, T)$  such that  $\lambda \leq \delta$ , then there exists a fuzzy residual set  $\mu$  in  $(X, T)$  such that  $\lambda \leq \mu$ .

**Proof:** Let  $\lambda$  be a fuzzy closed set and  $\delta$  be a fuzzy open set in  $(X, T)$  such that  $\lambda \leq \delta$ . Since  $(X, T)$  is a fuzzy Baire - separated space, by Proposition 3.1, there exists a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $\lambda \leq \eta \leq \delta$ . Then,  $\eta = \alpha \wedge \mu$ , where  $\alpha$  is a fuzzy open set and  $\mu$  is a fuzzy residual set in  $X$ . Now  $\alpha \wedge \mu \leq \mu$ , implies that  $\eta \leq \mu$  and  $\lambda \leq \eta \leq \mu$ . Thus, for a fuzzy closed set  $\lambda$  in  $(X, T)$ , there exists a fuzzy residual set  $\mu$  in  $(X, T)$  such that  $\lambda \leq \mu$ , in  $(X, T)$ .

*It is observed that fuzzy nowhere dense sets are fuzzy closed sets in fuzzy topological spaces [11] and under what conditions does a fuzzy Baire - separated space has a fuzzy closed set with fuzzy nowhere denseness ?*

**Proposition 3.5:** If  $\lambda$  is a fuzzy closed set and  $\delta$  is a fuzzy open set such that  $\lambda \leq \delta$  in a fuzzy Baire - separated space  $(X, T)$  in which fuzzy first category sets are fuzzy dense, then  $\lambda$  is a fuzzy nowhere dense set in  $X$ .

**Proof:** Let  $\lambda$  be a fuzzy closed set and  $\delta$  be a fuzzy open set in  $(X, T)$  such that  $\lambda \leq \delta$ . Since  $(X, T)$  is a fuzzy Baire - separated space, by Proposition 3.4, there exists a fuzzy residual set  $\mu$  in  $(X, T)$  such that  $\lambda \leq \mu$ . Then,  $\text{int}(\lambda) \leq \text{int}(\mu)$ . Now,  $1 - \mu$  is a fuzzy first category set in  $(X, T)$  and by hypothesis,  $\text{cl}(1 - \mu) = 1$ . By Lemma 2.1,  $1 - \text{int}(\mu) = \text{cl}(1 - \mu) = 1$  and it follows that  $\text{int}(\mu) = 0$  and then  $\text{int}(\lambda) \leq 0$ , in  $(X, T)$ . That is,  $\text{int}(\lambda) = 0$ . Since  $\lambda$  is a fuzzy closed set in  $X$ ,  $\text{cl}(\lambda) = \lambda$  and  $\text{int}(\text{cl}(\lambda)) = \text{int}(\lambda) = 0$ . Hence the fuzzy closed set  $\lambda$  is a fuzzy nowhere dense set in  $X$ .

**Proposition 3.6:** If  $\lambda$  is a fuzzy closed set and  $\delta$  is a fuzzy open set in a fuzzy Baire - separated space  $(X, T)$  such that  $\lambda \leq \delta$ , then there exists a fuzzy  $G_\delta$  - set  $\beta$  in  $(X, T)$  such that  $\beta \leq \delta$ .

**Proof:** Let  $\lambda$  be a fuzzy closed set and  $\delta$  be a fuzzy open set in  $(X, T)$  such that  $\lambda \leq \delta$ . Since  $(X, T)$  is a fuzzy Baire - separated space, by Proposition 3.1, there exists a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $\lambda \leq \eta \leq \delta$ . Now  $\eta$  being a fuzzy Baire set in  $(X, T)$ ,  $\eta = \alpha \wedge \mu$ , where  $\alpha$  is a fuzzy open set and  $\mu$  is a fuzzy residual set in  $(X, T)$ . Then,  $\lambda \leq \alpha \wedge \mu \leq \delta$ . By Theorem 2.4, for the fuzzy residual set  $\mu$  in  $X$  there exists a fuzzy  $G_\delta$  - set  $\gamma$  in  $(X, T)$  such that  $\gamma \leq \mu$ . Then,  $\alpha \wedge \gamma \leq \alpha \wedge \mu = \eta$ , in  $(X, T)$ . Let  $\beta = \alpha \wedge \gamma$ . Then,  $\beta$  is a fuzzy  $G_\delta$  - set in  $(X, T)$  such that  $\beta \leq \eta \leq \delta$ . Hence, for the fuzzy open set  $\delta$ , there exists a fuzzy  $G_\delta$  - set  $\beta$  such that  $\beta \leq \delta$ , in  $(X, T)$ .

**Corollary 3.2** If  $\mu_1$  and  $\mu_2$  are disjoint fuzzy closed sets in a fuzzy Baire - separated space  $(X, T)$ , then there exists a fuzzy  $G_\delta$  - set  $\eta$  in  $(X, T)$  such that  $\eta \leq 1 - \mu_2$  and  $\eta \leq 1 - \mu_1$ .

**Proof:** Let  $\mu_1$  and  $\mu_2$  be disjoint fuzzy closed sets in  $X$ . Then,  $\mu_1 \wedge \mu_2 = 0$ . This implies that  $\mu_1 \leq 1 - \mu_2$  and  $1 - \mu_2$  is a fuzzy open set in  $(X, T)$ . Since  $(X, T)$  is a fuzzy Baire - separated space, by Proposition 3.6, there exists a fuzzy  $G_\delta$  - set  $\eta$  in  $(X, T)$  such that  $\eta \leq 1 - \mu_2$ . Also  $\mu_1 \wedge \mu_2 = 0$ , implies that  $\mu_2 \leq 1 - \mu_1$  and then it follows that there exists a fuzzy  $G_\delta$  - set  $\eta$  in  $(X, T)$  such that  $\eta \leq 1 - \mu_1$ .

**Proposition 3.7:** If a fuzzy set  $\lambda$  is a fuzzy P-set such that  $\lambda \leq 1 - \mu$ , where  $\mu$  is a fuzzy  $F_\sigma$  - set in a fuzzy Baire - separated space  $(X, T)$ , then there exists a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $\lambda \leq \eta \leq 1 - \text{int}(\mu)$ .

**Proof:** Let  $\lambda$  be a fuzzy P-set such that  $\lambda \leq 1 - \mu$ , where  $\mu$  is a fuzzy  $F_\sigma$  - set in  $(X, T)$ . Then, by Theorem 2.5, there exists a fuzzy open set  $\delta$  in  $(X, T)$  such that  $\lambda \leq \delta \leq 1 - \text{int}(\mu)$ . Since  $\lambda$  is a fuzzy P-set in  $X$ ,  $\lambda$  is a fuzzy closed set and  $\lambda \leq \delta$ , in  $(X, T)$ . Since  $(X, T)$  is a fuzzy Baire - separated space, by Proposition 3.1, there exists a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $\lambda \leq \eta \leq \delta$ . This implies that  $\lambda \leq \eta \leq \delta \leq 1 - \text{int}(\mu)$  and hence  $\lambda \leq \eta \leq 1 - \text{int}(\mu)$ , in  $(X, T)$ .

*The following proposition gives a condition for a fuzzy topological space to become a fuzzy Baire - separated space.*

**Proposition 3.8:** If  $\mu_1 \leq (1 - \mu_2) \wedge (1 - \alpha)$ , for each pair of fuzzy closed sets  $\mu_1$  and  $\mu_2$  and for a fuzzy  $\sigma$  - nowhere dense set  $\alpha$ , in a fuzzy topological space  $(X, T)$ , then  $(X, T)$  is a fuzzy Baire - separated space.

**Proof:** Let  $\mu_1$  and  $\mu_2$  be a pair of fuzzy closed sets such that  $\mu_1 \leq (1 - \mu_2) \wedge (1 - \alpha)$ , where  $\alpha$  is a fuzzy  $\sigma$  - nowhere dense set in  $(X, T)$ . Now  $(1 - \mu_2) \wedge (1 - \alpha) \leq (1 - \mu_2)$  implies that  $\mu_1 \leq (1 - \mu_2)$ . Since  $\mu_2$  is a fuzzy closed set,  $1 - \mu_2$  is a fuzzy open set in  $(X, T)$ . Let  $\eta = (1 - \mu_2) \wedge (1 - \alpha)$ . Then, by Theorem 2.7,  $\eta$  is a fuzzy Baire set in  $(X, T)$ . Thus, for a pair of fuzzy closed sets  $\mu_1$  and  $\mu_2$  in  $(X, T)$  such that  $\mu_1 \leq 1 - \mu_2$ ,

the existence of a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $\mu_1 \leq \eta \leq 1 - \mu_2$ , implies that  $(X, T)$  is a fuzzy Baire - separated space.

**Corollary 3.3:** If  $\mu_1 \leq (1 - \mu_2) \wedge \beta$ , for each pair of fuzzy closed sets  $\mu_1$  and  $\mu_2$  and for a fuzzy  $G_\delta$  - set  $\beta$  such that  $\text{cl}(\beta) = 1$ , in a fuzzy topological space  $(X, T)$ , then  $(X, T)$  is a fuzzy Baire - separated space.

**Proof:** Now  $\beta = 1 - (1 - \beta)$ , where  $1 - \beta$  is a fuzzy  $F_\sigma$  - set in  $(X, T)$  and  $\text{int}(1 - \beta) = 1 - \text{cl}(\beta) = 1 - 1 = 0$ . Then,  $1 - \beta$  is a fuzzy  $\sigma$  - nowhere dense set in  $(X, T)$  and  $\mu_1 \leq (1 - \mu_2) \wedge (1 - (1 - \beta))$ . By Proposition 3.8,  $(X, T)$  is a fuzzy Baire - separated space.

**Proposition 3.9:** If  $\mu_1$  and  $\mu_2$  are disjoint fuzzy closed sets in a fuzzy Baire - separated space  $(X, T)$ , then there exists a fuzzy residual set  $\beta$  and a fuzzy first category set  $\gamma$  in  $(X, T)$  such that  $\mu_1 \leq \beta$  and  $\mu_2 \leq \alpha \vee \gamma$ , where  $\alpha$  is a fuzzy closed set in  $(X, T)$ .



Proof: Let  $\mu_1$  and  $\mu_2$  be disjoint fuzzy closed sets in  $(X, T)$ . Then,  $\mu_1 \wedge \mu_2 = 0$ . This implies that  $\mu_1 \leq 1 - \mu_2$  and  $1 - \mu_2$  is a fuzzy open set in  $(X, T)$ . Then, by Proposition 3.1, there exists a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $\mu_1 \leq \eta \leq 1 - \mu_2$ . Now  $\eta \leq 1 - \mu_2$ , implies that  $\mu_2 \leq 1 - \eta$  in  $(X, T)$ . Since  $\eta$  is a fuzzy Baire set,  $\eta = \lambda \wedge \delta$ , where  $\lambda$  is a fuzzy open set and  $\delta$  is a fuzzy residual set in  $(X, T)$  and thus  $\mu_2 \leq 1 - (\lambda \wedge \delta) = (1 - \lambda) \vee (1 - \delta)$ . Let  $\alpha = 1 - \lambda$  and  $\gamma = 1 - \delta$ . Then,  $\alpha$  is a fuzzy closed set and  $\gamma$  is a fuzzy first category set in  $(X, T)$ . By Proposition 3.4, for the fuzzy closed set  $\mu_1$ , there exists a fuzzy residual set  $\beta$  in  $(X, T)$  such that  $\mu_1 \leq \beta$ . Hence, for the disjoint fuzzy closed sets  $\mu_1$  and  $\mu_2$ , there exists a fuzzy residual set  $\beta$  and a fuzzy first category set  $\gamma$  in  $(X, T)$  such that  $\mu_1 \leq \beta$  and  $\mu_2 \leq \alpha \vee \gamma$ , where  $\alpha$  is a fuzzy closed set in  $(X, T)$ .

Proposition 3.10: If  $\mu_1$  and  $\mu_2$  are disjoint fuzzy closed sets in a fuzzy Baire - separated space  $(X, T)$ , then there exists a fuzzy residual set  $\beta$  and a fuzzy  $F_\sigma$ -set  $\delta$  such that  $\mu_1 \leq \beta$  and  $\mu_2 \leq \delta$ , in  $(X, T)$ .

Proof: Let  $\mu_1$  and  $\mu_2$  be disjoint fuzzy closed sets in  $(X, T)$ . Since  $(X, T)$  is a fuzzy Baire- separated space, by Proposition 3.9, there exists a fuzzy residual set  $\beta$  and a fuzzy first category set  $\gamma$  in  $(X, T)$  such that  $\mu_1 \leq \beta$  and  $\mu_2 \leq \alpha \vee \gamma$ , where  $\alpha$  is a fuzzy closed set in  $(X, T)$ . By Theorem 2.11, for the fuzzy first category set  $\gamma$ , there is a fuzzy  $F_\sigma$ -set  $\theta$  in  $(X, T)$  such that  $\gamma \leq \theta$  and then  $\alpha \vee \gamma \leq \alpha \vee \theta$ . Since  $\alpha$  is a fuzzy closed set,  $\alpha \vee \theta$  is a fuzzy  $F_\sigma$ -set in  $(X, T)$ . Let  $\delta = \alpha \vee \theta$ . Hence, for the disjoint fuzzy closed sets  $\mu_1$  and  $\mu_2$ , there exists a fuzzy residual set  $\beta$  and a fuzzy  $F_\sigma$ -set  $\delta$  such that  $\mu_1 \leq \beta$  and  $\mu_2 \leq \delta$ , in  $(X, T)$ .

Proposition 3.11: If the fuzzy sets  $(\lambda_i)$ 's ( $i = 1$  to  $\infty$ ) are fuzzy closed sets and  $(\delta_i)$ 's ( $i = 1$  to  $\infty$ ) are fuzzy open sets such that  $\lambda_i \leq \delta_i$  in a fuzzy Baire - separated space  $(X, T)$  in which fuzzy first category sets are fuzzy dense, then  $\bigvee_{i=1}^{\infty} (\delta_i)$  is a fuzzy open and fuzzy dense set in  $(X, T)$ .

Proof: Let  $(\lambda_i)$ 's ( $i = 1$  to  $\infty$ ) be fuzzy closed sets and  $(\delta_i)$ 's ( $i = 1$  to  $\infty$ ) be fuzzy open sets in a fuzzy Baire - separated space  $(X, T)$  such that  $\lambda_i \leq \delta_i$ . This implies that  $\bigvee_{i=1}^{\infty} (\lambda_i) \leq \bigvee_{i=1}^{\infty} (\delta_i)$  and  $\text{cl}[\bigvee_{i=1}^{\infty} (\lambda_i)] \leq \text{cl}[\bigvee_{i=1}^{\infty} (\delta_i)]$ , in  $(X, T)$ . Since  $(X, T)$  is a fuzzy Baire - separated space in which fuzzy first category sets are fuzzy dense, by Proposition 3.5,  $(\lambda_i)$ 's are fuzzy nowhere dense sets in  $(X, T)$ . Then,  $\bigvee_{i=1}^{\infty} (\lambda_i)$  is a fuzzy first category set in  $(X, T)$ . By hypothesis,  $\text{cl}[\bigvee_{i=1}^{\infty} (\lambda_i)] = 1$ . This implies that  $\text{cl}[\bigvee_{i=1}^{\infty} (\delta_i)] = 1$  and thus  $\bigvee_{i=1}^{\infty} (\delta_i)$  is a fuzzy open and fuzzy dense set in  $(X, T)$ .

Corollary 3.4: If the fuzzy sets  $(\lambda_i)$ 's ( $i = 1$  to  $\infty$ ) are fuzzy closed sets and  $(\delta_i)$ 's ( $i = 1$  to  $\infty$ ) are fuzzy open sets such that  $\lambda_i \leq \delta_i$  in a fuzzy Baire - separated space  $(X, T)$  in which fuzzy first category sets are fuzzy dense, then there exists a fuzzy first category and fuzzy  $F_\sigma$ -set  $\lambda$  and a fuzzy open and fuzzy dense set  $\delta$  in  $(X, T)$  such that  $\lambda_i \leq \lambda$ ,  $\delta_i \leq \delta$  and  $\lambda \leq \delta$ .

Proof: Let  $(\lambda_i)$ 's ( $i = 1$  to  $\infty$ ) be fuzzy closed sets and  $(\delta_i)$ 's ( $i = 1$  to  $\infty$ ) be fuzzy open sets in a fuzzy Baire - separated space  $(X, T)$  such that  $\lambda_i \leq \delta_i$ . Then, by Proposition 3.11,  $\bigvee_{i=1}^{\infty} (\delta_i)$  is a fuzzy open and fuzzy dense set in  $(X, T)$ . Also  $\bigvee_{i=1}^{\infty} (\lambda_i)$  is a fuzzy first category set in  $(X, T)$ . Since  $(\lambda_i)$ 's are fuzzy closed sets,  $\bigvee_{i=1}^{\infty} (\lambda_i)$  is a fuzzy  $F_\sigma$ -set in  $(X, T)$ . Let  $\lambda = \bigvee_{i=1}^{\infty} (\lambda_i)$  and  $\delta = \bigvee_{i=1}^{\infty} (\delta_i)$ . Hence, there exists a fuzzy first category and fuzzy  $F_\sigma$ -set  $\lambda$  and a fuzzy open and fuzzy dense set  $\delta$  in  $(X, T)$  such that  $\lambda_i \leq \lambda$ ,  $\delta_i \leq \delta$  and  $\lambda \leq \delta$ .

The following proposition gives a condition for a fuzzy topological space to become a fuzzy Baire - separated space.

Proposition 3.12: If  $\lambda \leq \delta \wedge \mu$ , for each fuzzy closed set  $\lambda$  and each fuzzy open set  $\delta$ , where  $\mu$  is a fuzzy residual set in a fuzzy topological space  $(X, T)$ , then  $(X, T)$  is a fuzzy Baire - separated space.

Proof: Suppose that  $\lambda \leq \delta \wedge \mu$ , for each fuzzy closed set  $\lambda$  and each fuzzy open set  $\delta$ , where  $\mu$  is a fuzzy residual set in  $(X, T)$ . Now  $\delta \wedge \mu \leq \delta$ , implies that  $\lambda \leq \delta \wedge \mu \leq \delta$ . Let  $\eta = \delta \wedge \mu$ . Then,  $\eta$  is a fuzzy Baire set in  $(X, T)$ . Hence  $(X, T)$  is a fuzzy Baire - separated space.

Corollary 3.5: If  $\lambda \vee \eta \leq \delta$ , for each fuzzy closed set  $\lambda$  and each fuzzy open set  $\delta$ , where  $\eta$  is a fuzzy first category set in a fuzzy topological space  $(X, T)$ , then  $(X, T)$  is a fuzzy Baire - separated space.

## 4. Fuzzy Baire-Separated Spaces and Other Fuzzy Topological Spaces

Proposition 4.1: If  $\lambda$  is a fuzzy closed set and  $\delta$  is a fuzzy open set in a fuzzy Baire - separated and fuzzy globally disconnected space  $(X, T)$  with  $\lambda \leq \delta$ , then there exists a fuzzy  $G_\delta$ -set  $\eta$  in  $(X, T)$  such that  $\lambda \leq \eta \leq \delta$ .

Proof: Let  $\lambda$  be a fuzzy closed set and  $\delta$  be a fuzzy open set in  $(X, T)$  with  $\lambda \leq \delta$ . Since  $(X, T)$  is a fuzzy Baire - separated space, by Proposition 3.1, there exists a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $\lambda \leq \eta \leq \delta$ . Also since  $(X, T)$  is a fuzzy globally disconnected space, by Theorem 2.1, the fuzzy Baire set  $\eta$  is a fuzzy  $G_\delta$ -set in  $X$ . Hence, for a fuzzy closed set  $\lambda$  and a fuzzy open set  $\delta$  with  $\lambda \leq \delta$ , there exists a fuzzy  $G_\delta$ -set  $\eta$  in  $X$ , such that  $\lambda \leq \eta \leq \delta$ .

Corollary 4.1: If  $\mu_1$  and  $\mu_2$  are disjoint fuzzy closed sets in a fuzzy Baire-separated and fuzzy globally disconnected space  $(X, T)$ , then there exists a fuzzy  $G_\delta$ -set  $\eta$  in  $(X, T)$  such that  $\mu_1 \leq \eta \leq 1 - \mu_2$ .

Proof: Let  $\mu_1$  and  $\mu_2$  be disjoint fuzzy closed sets in  $(X, T)$ . Then,  $\mu_1 \wedge \mu_2 = 0$ . This implies that  $\mu_1 \leq 1 - \mu_2$  and  $1 - \mu_2$  is a fuzzy open set in  $(X, T)$ . Since  $(X, T)$  is a fuzzy Baire - separated and fuzzy globally disconnected space, by Proposition 4.1, there exists a fuzzy  $G_\delta$ -set  $\eta$  in  $(X, T)$  such that  $\mu_1 \leq \eta \leq 1 - \mu_2$ .

Proposition 4.2: If  $\lambda$  is a fuzzy closed set and  $\delta$  is a fuzzy

open set with  $\lambda \leq \delta$  in a fuzzy Baire -separated and fuzzy submaximal space  $(X, T)$ , then there exists a fuzzy  $G_\delta$  -set  $\eta$  in  $(X, T)$  such that  $\lambda \leq \eta \leq \delta$ .

Proof: Let  $\lambda$  be a fuzzy closed set and  $\delta$  be a fuzzy open set in  $(X, T)$  with  $\lambda \leq \delta$ . Since  $(X, T)$  is a fuzzy Baire - separated space, by Proposition 3.1, there exists a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $\lambda \leq \eta \leq \delta$ . Also since  $(X, T)$  is a fuzzy submaximal space, by Theorem 2.2, the fuzzy Baire set  $\eta$  is a fuzzy  $G_\delta$  -set in  $(X, T)$ . Hence, for a fuzzy closed set  $\lambda$  and a fuzzy open set  $\delta$  with  $\lambda \leq \delta$ , there exists a fuzzy  $G_\delta$  -set  $\eta$  in  $X$  such that  $\lambda \leq \eta \leq \delta$ .

Corollary 4.2: If  $\mu_1$  and  $\mu_2$  are disjoint fuzzy closed sets in a fuzzy Baire-separated and fuzzy submaximal space  $(X, T)$ , then there exists a fuzzy  $G_\delta$  -set  $\eta$  in  $(X, T)$  such that  $\mu_1 \leq \eta \leq 1 - \mu_2$ .

Corollary 4.3: If  $\alpha$  is a fuzzy open set and  $\beta$  is a fuzzy closed set with  $\beta \leq \alpha$  in a fuzzy Baire-separated and fuzzy submaximal space  $(X, T)$ , then there exists a fuzzy  $F_\sigma$ -set  $\delta$  in  $(X, T)$  such that  $\beta \leq \delta \leq \alpha$ .

Proof: Let  $\alpha$  be a fuzzy open set and  $\beta$  be a fuzzy closed set in  $(X, T)$  with  $\beta \leq \alpha$ . Then,  $1 - \alpha \leq 1 - \beta$ , where  $1 - \alpha$  is a fuzzy closed set and  $1 - \beta$  is a fuzzy open set in the fuzzy Baire - separated and fuzzy submaximal space  $(X, T)$ . By Proposition 4.2, there exists a fuzzy  $G_\delta$  -set  $\eta$  in  $(X, T)$  such that  $1 - \alpha \leq \eta \leq 1 - \beta$ . Then, it follows that  $\beta \leq 1 - \eta \leq \alpha$ . Let  $\delta = 1 - \eta$ . Then,  $\delta$  is a fuzzy  $F_\sigma$  -set in  $(X, T)$  and  $\beta \leq \delta \leq \alpha$ , in  $(X, T)$ .

*It is observed that fuzzy nowhere dense sets are fuzzy closed sets in fuzzy nodec spaces [11] and subsequently a question will arise. Which fuzzy topological spaces [other than fuzzy hyperconnected spaces [23], fuzzy globally disconnected spaces] have fuzzy closed sets with fuzzy nowhere denseness? The following propositions give the answers for this question.*

Proposition 4.3: If  $\lambda$  is a fuzzy closed set and  $\delta$  is a fuzzy open set with  $\lambda \leq \delta$  in a fuzzy Baire-separated and fuzzy extraresolvable space  $(X, T)$ , then the fuzzy closed set  $\lambda$  is a fuzzy nowhere dense set in  $(X, T)$ .

Proof: Let  $\lambda$  be a fuzzy closed set and  $\delta$  be a fuzzy open set in  $(X, T)$  with  $\lambda \leq \delta$ . Since  $(X, T)$  is a fuzzy Baire -separated space, by Proposition 3.4, there exists a fuzzy residual set  $\mu$  in  $(X, T)$  such that  $\lambda \leq \mu$ . Then,  $\text{int}(\lambda) \leq \text{int}(\mu)$ , in  $(X, T)$ . Also since  $(X, T)$  is a fuzzy extra-resolvable space, by Theorem 2.3,  $\text{int}(\mu) = 0$  and then  $\text{int}(\lambda) \leq 0$ . That is,  $\text{int}(\lambda) = 0$ , in  $(X, T)$ . Since  $\lambda$  is a fuzzy closed set,  $\text{cl}(\lambda) = \lambda$  and  $\text{int} \text{cl}(\lambda) = \text{int}(\lambda) = 0$ . Hence the fuzzy closed set  $\lambda$  is a fuzzy nowhere dense set in  $(X, T)$ .

Proposition 4.4: If  $\lambda$  is a fuzzy closed set and  $\delta$  is a fuzzy open set with  $\lambda \leq \delta$  in a fuzzy Baire - separated and fuzzy strongly Baire space  $(X, T)$ , then the fuzzy closed set  $\lambda$  is a fuzzy nowhere dense set in  $(X, T)$ .

Proof: Let  $\lambda$  be a fuzzy closed set and  $\delta$  be a fuzzy open set in  $(X, T)$  with  $\lambda \leq \delta$ . Since  $(X, T)$  is a fuzzy Baire - sepa-

rated space, by Proposition 3.4, there exists a fuzzy residual set  $\mu$  in  $(X, T)$  such that  $\lambda \leq \mu$ . Then,  $\text{int}(\lambda) \leq \text{int}(\mu)$ , in  $(X, T)$ . Also since  $(X, T)$  is a fuzzy strongly Baire space, by Theorem 2.9,  $\text{int}(\mu) = 0$  and then  $\text{int}(\lambda) \leq 0$ . That is,  $\text{int}(\lambda) = 0$ , in  $(X, T)$ . Since  $\lambda$  is a fuzzy closed set in  $X$ ,  $\text{cl}(\lambda) = \lambda$  and  $\text{int} \text{cl}(\lambda) = \text{int}(\lambda) = 0$ . Hence the fuzzy closed set  $\lambda$  is a fuzzy nowhere dense set in  $(X, T)$ .

Corollary 4.4: If  $\mu_1$  and  $\mu_2$  are disjoint fuzzy closed sets in a fuzzy Baire - separated and fuzzy strongly Baire space  $(X, T)$ , then  $\mu_1$  and  $\mu_2$  are fuzzy nowhere dense sets in  $(X, T)$ .

Proof: Let  $\mu_1$  and  $\mu_2$  be disjoint fuzzy closed sets in  $(X, T)$ . Then,  $\mu_1 \wedge \mu_2 = 0$ . This implies that  $\mu_1 \leq 1 - \mu_2$  and  $1 - \mu_2$  is a fuzzy open set in  $(X, T)$ . Since  $(X, T)$  is a fuzzy Baire - separated space, by Proposition 3.4, there exists a fuzzy residual set  $\delta$  in  $X$  such that  $\mu_1 \leq \delta$ . Then,  $\text{int}(\mu_1) \leq \text{int}(\delta)$ , in  $(X, T)$ . Also since  $(X, T)$  is a fuzzy strongly Baire space, by Theorem 2.9,  $\text{int}(\delta) = 0$  and then  $\text{int}(\mu_1) \leq 0$ . That is,  $\text{int}(\mu_1) = 0$ , in  $(X, T)$ . Since  $\mu_1$  is a fuzzy closed set in  $X$ ,  $\text{cl}(\mu_1) = \mu_1$  and  $\text{int} \text{cl}(\mu_1) = \text{int}(\mu_1) = 0$ . Hence  $\mu_1$  is a fuzzy nowhere dense set in  $(X, T)$ . Also  $\mu_1 \wedge \mu_2 = 0$ , implies that  $\mu_2 \leq 1 - \mu_1$  and then it follows that  $\mu_2$  is also a fuzzy nowhere dense set in  $(X, T)$ .

Corollary 4.5: If  $\mu_1$  and  $\mu_2$  are disjoint fuzzy closed sets in a fuzzy Baire -separated and fuzzy extraresolvable space  $(X, T)$ , then  $\mu_1$  and  $\mu_2$  are fuzzy nowhere dense sets in  $(X, T)$ .

Proposition 4.5: If  $\lambda$  is a fuzzy closed set and  $\delta$  is a fuzzy open set in a fuzzy Baire -separated and fuzzy second category (but not fuzzy Baire) and fuzzy hyper-connected space  $(X, T)$  such that  $\lambda \leq \delta$ , then the fuzzy closed set  $\lambda$  is a fuzzy nowhere dense set in  $(X, T)$ .

Proof: Let  $\lambda$  be a fuzzy closed set and  $\delta$  be a fuzzy open set in  $(X, T)$  such that  $\lambda \leq \delta$ . Since  $(X, T)$  is a fuzzy Baire -separated space, by Proposition 3.4, there exists a fuzzy residual set  $\mu$  in  $(X, T)$  such that  $\lambda \leq \mu$ . Then,  $\text{int}(\lambda) \leq \text{int}(\mu)$ , in  $(X, T)$ . Also since  $(X, T)$  is a fuzzy second category (but not fuzzy Baire) and fuzzy hyper-connected space, by Theorem 2.12, the fuzzy residual set  $\mu$  is a fuzzy nowhere dense set in  $(X, T)$ . Then,  $\text{int} \text{cl}(\mu) = 0$  and  $\text{int}(\mu) \leq \text{int} \text{cl}(\mu)$  implies that  $\text{int}(\mu) = 0$  and then  $\text{int}(\lambda) = 0$ , in  $(X, T)$ . Since  $\lambda$  is a fuzzy closed set,  $\text{cl}(\lambda) = \lambda$  and  $\text{int} \text{cl}(\lambda) = \text{int}(\lambda) = 0$ . Hence the fuzzy closed set  $\lambda$  is a fuzzy nowhere dense set in  $(X, T)$ .

*The following proposition shows that fuzzy ultra-connected spaces, are not fuzzy Baire - separated spaces.*

Proposition 4.6: If a fuzzy topological space  $(X, T)$  is a fuzzy ultraconnected space, then  $(X, T)$  is not a fuzzy Baire - separated space.

Proof: Let  $\mu_1$  and  $\mu_2$  be a pair of fuzzy closed sets in  $(X, T)$ . Since  $(X, T)$  is a fuzzy ultraconnected space,  $\mu_1 \not\leq 1 - \mu_2$  in  $(X, T)$ . Then, it is not possible to find a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $\mu_1 \leq \eta \leq 1 - \mu_2$ . Hence  $(X, T)$  is not a fuzzy Baire - separated space.

*The following proposition gives a condition under which fuzzy weakly Baire spaces become fuzzy Baire - separated spaces.*

Proposition 4.7: If  $\mu_1 \leq (1 - \mu_2) \wedge (1 - \alpha)$ , where  $\alpha$  is a fuzzy  $\sigma$ -boundary set and  $\mu_1$  and  $\mu_2$  are fuzzy closed sets in a fuzzy weakly Baire space  $(X, T)$ , then  $(X, T)$  is a fuzzy Baire - separated space.

Proof: Let  $\mu_1$  and  $\mu_2$  be a pair of fuzzy closed sets such that  $\mu_1 \leq (1 - \mu_2) \wedge (1 - \alpha)$ , where  $\alpha$  is a fuzzy  $\sigma$ -boundary set in  $(X, T)$ . Since  $(X, T)$  is a fuzzy weakly Baire space, by Theorem 2.8, the fuzzy  $\sigma$ -boundary set  $\alpha$  is a fuzzy  $\sigma$ -nowhere dense set in  $X$ . Thus,  $\mu_1 \leq (1 - \mu_2) \wedge (1 - \alpha)$ , where  $\alpha$  is a fuzzy  $\sigma$ -nowhere dense set in  $(X, T)$ . Then, by Proposition 3.8,  $(X, T)$  is a fuzzy Baire - separated space.

Proposition 4.8: If  $\mu_1 \leq 1 - \mu_2$ , for any two non - zero fuzzy sets  $\mu_1$  and  $\mu_2$  in a fuzzy perfectly disconnected and fuzzy Baire - separated space  $(X, T)$ , then there exists a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $\mu_1 \leq \eta \leq 1 - \mu_2$ .

Proof: Let  $\mu_1$  and  $\mu_2$  be any two fuzzy sets defined on  $X$  such that  $\mu_1 \leq 1 - \mu_2$ . Since  $(X, T)$  is a fuzzy perfectly disconnected space,  $\mu_1 \leq 1 - \mu_2$  implies that  $\text{cl}(\mu_1) \leq 1 - \text{cl}(\mu_2)$ . By Theorem 2.10,  $\text{cl}(\mu_1) \neq 1$  and  $\text{cl}(\mu_2) \neq 1$ , in  $(X, T)$ . Also since  $(X, T)$  is a fuzzy Baire - separated space, for the fuzzy closed sets  $\text{cl}(\mu_1)$  and  $\text{cl}(\mu_2)$ , there exists a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $\text{cl}(\mu_1) \leq \eta \leq 1 - \text{cl}(\mu_2)$ . Now  $\mu_1 \leq \text{cl}(\mu_1) \leq \eta \leq 1 - \text{cl}(\mu_2) \leq 1 - \mu_2$ , implies that  $\mu_1 \leq \eta \leq 1 - \mu_2$ , in  $X$ .

Proposition 4.9: If the fuzzy sets  $\mu_1$  and  $\mu_2$  are disjoint fuzzy  $F_\sigma$  -sets in a fuzzy  $F'$ - space and fuzzy Baire - separated space  $(X, T)$ , then there exists a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $\mu_1 \leq \eta \leq 1 - \mu_2$ .

Proof: Let  $\mu_1$  and  $\mu_2$  be disjoint fuzzy  $F_\sigma$ -sets in  $(X, T)$ . Then,  $\mu_1 \wedge \mu_2 = 0$ . This implies that  $\mu_1 \leq 1 - \mu_2$ . Since  $(X, T)$  is a fuzzy  $F'$ - space,  $\mu_1 \leq 1 - \mu_2$  implies that  $\text{cl}(\mu_1) \leq 1 - \text{cl}(\mu_2)$ . Also since  $(X, T)$  is a fuzzy Baire-separated space, for the fuzzy closed sets  $\text{cl}(\mu_1)$  and  $\text{cl}(\mu_2)$ , there exists a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $\text{cl}(\mu_1) \leq \eta \leq 1 - \text{cl}(\mu_2)$ . Now  $\mu_1 \leq \text{cl}(\mu_1) \leq \eta \leq 1 - \text{cl}(\mu_2) \leq 1 - \mu_2$ , implies that  $\mu_1 \leq \eta \leq 1 - \mu_2$ , in  $(X, T)$ .

Proposition 4.10: If the fuzzy sets  $\mu_1$  and  $\mu_2$  are disjoint fuzzy closed sets in a fuzzy Baire - separated and fuzzy globally disconnected space  $(X, T)$ , then there exists a fuzzy  $G_\delta$  - set  $\eta$  and a fuzzy  $F_\sigma$ -set  $\delta$  in  $(X, T)$  such that  $\mu_1 \leq \eta$  and  $\mu_2 \leq \delta$ .

Proof: Let  $\mu_1$  and  $\mu_2$  be disjoint fuzzy closed sets in  $(X, T)$ . Since  $(X, T)$  is a fuzzy Baire - separated space, by Corollary 4.1, there exists a fuzzy  $G_\delta$  -set  $\eta$  in  $(X, T)$  such that  $\mu_1 \leq \eta \leq 1 - \mu_2$ . Now  $\eta \leq 1 - \mu_2$ , implies that  $\mu_2 \leq 1 - \eta$ , in  $(X, T)$ . Since  $\eta$  is a fuzzy  $G_\delta$  -set,  $1 - \eta$  is a fuzzy  $F_\sigma$ -set in  $(X, T)$ . Let  $\delta = 1 - \eta$ . Hence, for the disjoint fuzzy closed sets  $\mu_1$  and  $\mu_2$ , there exists a fuzzy  $G_\delta$  -set  $\eta$  and a fuzzy  $F_\sigma$  -set  $\delta$  in  $(X, T)$  such that  $\mu_1 \leq \eta$  and  $\mu_2 \leq \delta$ .

Proposition 4.11: If the fuzzy sets  $\mu_1$  and  $\mu_2$  are disjoint fuzzy closed sets in a fuzzy Baire -separated and fuzzy sub-maximal space  $(X, T)$ , then there exists a fuzzy  $G_\delta$  -set  $\eta$  and a fuzzy  $F_\sigma$  -set  $\delta$  in  $(X, T)$  such that  $\mu_1 \leq \eta$  and  $\mu_2 \leq \delta$ .

Proof: The proof follows from Corollary 4.2.

Proposition 4.12: If the fuzzy sets  $\mu_1$  and  $\mu_2$  are disjoint fuzzy closed sets in a fuzzy Baire -separated, fuzzy globally disconnected and weak fuzzy Oz-space  $(X, T)$ , then there exist fuzzy  $G_\delta$  -sets  $\eta$  and  $\theta$  in  $(X, T)$  such that  $\mu_1 \leq \eta$  and  $\mu_2 \leq \theta$ .

Proof: Let  $\mu_1$  and  $\mu_2$  be disjoint fuzzy closed sets in  $(X, T)$ . Since  $(X, T)$  is a fuzzy Baire - separated, fuzzy globally disconnected space, by Proposition 4.10, there exists a fuzzy  $G_\delta$  -set  $\eta$  and a fuzzy  $F_\sigma$  -set  $\delta$  in  $(X, T)$  such that  $\mu_1 \leq \eta$  and  $\mu_2 \leq \delta$ . Now  $\mu_2 \leq \delta$  implies that  $\text{cl}(\mu_2) \leq \text{cl}(\delta)$ , and then since  $\mu_2$  is a fuzzy closed set,  $\mu_2 = \text{cl}(\mu_2) \leq \text{cl}(\delta)$ , in  $(X, T)$ . Also since  $(X, T)$  is a weak fuzzy Oz-space, for the fuzzy  $F_\sigma$ -set  $\delta$ ,  $\text{cl}(\delta)$  is a fuzzy  $G_\delta$  -set in  $(X, T)$ . Let  $\theta = \text{cl}(\delta)$ . Hence, for the disjoint fuzzy closed sets  $\mu_1$  and  $\mu_2$ , there exist fuzzy  $G_\delta$  -sets  $\eta$  and  $\theta$  in  $(X, T)$  such that  $\mu_1 \leq \eta$  and  $\mu_2 \leq \theta$ .

The following proposition gives a condition under which fuzzy Baire - separated spaces become fuzzy semi normal spaces.

Proposition 4.13: If a fuzzy  $(X, T)$  is a fuzzy Baire -separated space in which fuzzy Baire sets are fuzzy regular open sets, then  $(X, T)$  is a fuzzy seminormal space.

Proof: Let  $\lambda$  be a fuzzy closed set and  $\delta$  be a fuzzy open set in  $(X, T)$  such that  $\lambda \leq \delta$ . Since  $(X, T)$  is a fuzzy Baire - separated space, by Proposition 3.1, there exists a fuzzy Baire set  $\eta$  in  $(X, T)$  such that  $\lambda \leq \eta \leq \delta$ . By hypothesis, the fuzzy Baire set  $\eta$  is a fuzzy regular open set in  $(X, T)$  and thus for the fuzzy closed set  $\lambda$  and the fuzzy open set  $\delta$  such that  $\lambda \leq \delta$ , there exists a fuzzy regular open set  $\eta$  in  $(X, T)$  such that  $\lambda \leq \eta \leq \delta$ . Hence it follows that  $(X, T)$  is a fuzzy semi-normal space.

Proposition 4.14: If a fuzzy  $(X, T)$  is a fuzzy Baire - separated space in which for each fuzzy Baire set  $\eta$ ,  $\eta = \text{int}(\gamma)$ , where  $(1 - \gamma) \in T$ , then  $(X, T)$  is a fuzzy semi normal space.

Proof: The proof follows from Lemma 2.1 and Proposition 4.13.

## 5. Conclusion

In this paper, the notion of fuzzy Baire-separated spaces is introduced in terms of fuzzy Baire sets. It is shown that fuzzy Baire sets lie between disjoint fuzzy  $P$ -sets and fuzzy  $F_\sigma$ -sets in a fuzzy Baire-separated space. Also it is obtained that fuzzy Baire sets which lie between fuzzy closed sets and fuzzy open sets in a fuzzy Baire-separated space are not fuzzy dense sets. The conditions under which fuzzy topological spaces become fuzzy Baire-separated spaces are identified. Fuzzy nowhere dense sets are fuzzy closed sets in fuzzy nodec spaces and subsequently the fuzzy topological spaces having fuzzy closed sets with fuzzy nowhere denseness are identified and it is shown that fuzzy ultraconnected spaces are not fuzzy Baire - separated spaces. The means, by which fuzzy weakly Baire spaces become fuzzy Baire -separated spaces and the fuzzy Baire-separated spaces become fuzzy semi-normal spaces are obtained. In the future work, the in-

ter-relations between fuzzy Baire spaces and Baire -separated spaces have to be explored.

## Conflicts of Interest

The authors declare no conflicts of interest.

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