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Research Article

About Absolute Convergence of Fourier Series of Almost Periodic Functions

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Abstract

The current stage of development of the theory of almost periodic functions is characterized by a desire for analysis and processing of a huge amount of accumulated scientific and practical material. The theory of almost periodic functions arose in the 20-30 s of the twentieth century; currently, extensive literature has accumulated on various issues of this theory. Long before the creation of the general theory of almost periodic functions, the outstanding Riga mathematician P. Bol drew attention to such functions. For functions of many variables $f(x_1, x_{2,}...x_p)$, Bol considered the corresponding multiple Fourier series and, in p-dimensional Euclidean space, a straight line passing through the origin: $x_1=a_1$ t, $x_2=a_2$ t,..., $x_p=a_pt$, where a_1, a_2, ..., a_p - some real, non-zero numbers. Considering the value of the function $f(x_1, x_{2,}...x_p)$ on this line, he obtains a function of one variable $\varphi(t) = f(a_1 t, a_2 t,...a_p t)$ and proves that this function is almost periodic. With some choice of numbers a_1, a_2, ..., a_p - it may happen that this function is periodic. However, if the numbers a_1, a_2, ..., a_p are linearly independent, then you can easily make sure that the function will not be a periodic function. Further development of the problem was carried out by the French mathematician E. Escalangon. However, the main significant drawback of the results of Bol and Escalangon was that from the very beginning, starting with the definition of almost-periodic functions, they introduced into consideration a fixed system of numbers a_1, a_2, ..., a_p associated with the almost-period (τ). This drawback was eliminated by the Danish mathematician G. Bohr, who developed in general terms the theory of continuous almost-periodic functions. Bohr's research in its methods was closely related to Bohl's research. However, Bohr did not impose restrictions such as Bohl's inequality in advance for the almost period. The results obtained by Bol and Bohr were based on the deep connection between almost periodic functions and periodic functions of many variables. The article examines the question of sufficient conditions for the absolute and uniform convergence of Fourier series of uniform almost periodic functions in the case when the Fourier exponents have a single limit point at zero, i.e. $\lambda_k \to 0$ $(k \to \infty)$. In this case, the Laplace transform is used for the first time as a structural characteristic.

Keywords

Almost Periodic Bohr Functions, Fourier Series, Spectrum Functions, Fourier Coefficients, Trigonometric Polynomials, Best Uniform Approximation, Limit Point at Zero, Laplace Transform

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1. Introduction

When studying the convergence of Fourier series of uniform almost-periodic functions, we are faced from the very beginning with a serious difficulty, namely that the Fourier exponents can lie densely everywhere, and therefore it is not clear in what order the terms of the Fourier series should be summed. In the case when the Fourier series converges absolutely, the question of the order of the terms of the Fourier series disappears.

Let $f(x)$ be a function integrable with degree p $(1 \leq p \leq \infty)$ on the interval $[-\pi;\pi]$ with norm

$$\|f(x)\|_p = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^p\,dx\right\}^{1/p} < \infty\ (1 \leq p < \infty),$$

and for $p = \infty$

$$\|f(x)\|_p = vrai\sup_{-\infty<x<\infty}|f(x)| < \infty,$$

and has a Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m\cos mx + b_m\sin mx),$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx,\ a_m = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos mx\,dx,$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin mx\,dx$$

Fourier coefficients of the function $f \in L_p[-\pi,\pi]$ (see, for example, [1-5]).

Definition 1 [2]. The function $f(x)$ is called - almost-periodic, or almost-periodic in the sense of Bezikovich $(p \geq 1)$, if

1. $f(x)$ измеримая и $|f(x)|^p$ интегрируема в смысле Лебега на любом конечном отрезке;
2. $D_{B_p}\{f(x)\} = \left\{\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}|f(x)|^p dx\right\}^{1/p} < \infty;$
3. There is a sequence of trigonometric sums

$$P_n(x) = \sum_{k=1}^{n}C_k\,exp(i\lambda_k x),$$

for which $lim_{n\to\infty}D_{B_p}\{f(x) - P_n(x)\} = 0$.

The space of such functions that satisfy all the conditions of definition 1. is called B_p – space, or Bozicevic space, in which the norm of the function is $f(x) \in B_p$ (p≥1) the value is assumed

$$\|f(x)\|_{B_p} = \left\{\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}|f(x)|^p dx\right\}^{\frac{1}{p}} < \infty.$$

As can be seen from definition 1., a sequence of numbers $\Lambda\{\lambda_n\}$ is associated with each function from the space B_p $(p \geq 1)$, which is the spectrum of this function. Under the spectrum $\Lambda\{\lambda_n\}$ for the function $f(x) \in B_p$ is understood as the set of its Fourier exponents, which can be used to match the Fourier series.

In the works of B. M. Levitan [5], E. A. Bredikhina [6, 7], A. S. Museliak [8], N. P. Kuptsov [9], A. G. Pritula [10], A. S. Jafarov and G. A. Mammadov [11], Yu. Kh. Khasanova, F. M. Talbakova [12-16] and others obtained some necessary and sufficient conditions for the absolute convergence of Fourier series of almost periodic in the sense of Bohr and Bezikovich functions.

J. Museliaka [8] showed that if the spectrum $\lambda_n \to \infty$ and $n^\alpha = O\{\lambda_n\}$, $n \to \infty, \alpha > 0$, then for the function $f(x) \in B_2$ the conditio

$$\sum_{n=1}^{\infty} n^{\frac{1-\frac{\beta}{2}}{\alpha}-1}\,\omega_1^\beta\left(f;\frac{1}{n}\right)_{\boldsymbol{B_2}} < \infty \quad (1)$$

for $0 < \beta < 2$ the series converges $\sum_{n=1}^{\infty}|A_n|^\beta < \infty$.

N. P. Kuptsov [9] showed that for functions $F(x) \in B$ condition (1) with $\alpha = 1$, $\beta = 1$with the change of quantity $\omega_1(f;\frac{1}{n})_{\boldsymbol{B_1}}$ on $\omega_2(f;\frac{1}{n})_{\boldsymbol{B_2}}$ ensures the validity of relation $\sum_{n=1}^{\infty}|A_n|^\beta < \infty$.

In the work of A. G. Prituly [10] prove that if $\lambda_n \to \infty, 0 < \beta < q, 2 \leq q < \infty, \gamma > 0$ condition is met

$$\sum_{\nu=1}^{\infty}\left(\frac{\lambda_{2^\nu}}{\lambda_{2^{\nu-1}}}\right)^\beta \omega_1^\beta\left(f;\frac{1}{\lambda_{2^\nu}}\right)_{B_p} 2^{\nu\left(\gamma+\frac{q-\beta}{q}\right)} < \infty, \text{That}$$

$$\sum_{n=1}^{\infty}|A_n|^\beta n^\beta < \infty.$$

In the case when $\in B_p$, $1 < \mathrm{p} \leq 2$, $\lambda_n \to 0$, A. S. Jafarova and G. A. Mamedova [11] established the convergence of the series $\sum_{n=1}^{\infty}|A_n|^\beta\varphi(n)$, under certain conditions on $\varphi(n)$. Instead of the continuity modulus they used the characteristic

$$\Omega(f;H;\delta;\theta) = \delta\min_x\left|\int_0^\infty \exp(-\delta\theta)f(x-t)\exp(i\theta t)dt\right|, \delta > 0, \theta \in R.$$

In the work of Yu. Kh. Khasanov [12] established some necessary and sufficient conditions for the absolute convergence of Fourier series of almost-periodic Besicovitch functions when the Fourier exponents have limit points at infinity or zero. The results of this work are analogues of some results from [13, 16] for the class of uniform almost periodic Bohr functions.

2. Main Results

The work examines sufficient conditions for the absolute and uniform convergence of Fourier series of functions that are almost periodic in the sense of Bohr.

Almost periodicity is a generalization of ordinary periodicity

To obtain guiding considerations for determining almost periodicity, consider the following example. Let

$$q(\mathrm{x}) = \cos \mathrm{x} + \cos\sqrt{2}\, x.$$

Each term in this sum is a periodic function but

the periods are incommensurable and therefore the sum is not periodic

function. However, it is easy to establish the existence of the function $q(\mathrm{x})$ as

called displacements or almost periods. This follows from one theorem

Kronecker, which, by the way, will be proven in the next paragraph

In particular, in our case, from Kronecker's theorem it follows that for

arbitrary, positive number 8 "those) there are integers

n_1 and n_2 and an arbitrarily large real number m, which are satisfactory allow inequalities

$$|\tau - 2\pi \mathrm{n}_1| < \delta, \quad \left|\sqrt{2}\tau - 2\pi \mathrm{n}_2\right| < \delta.$$

That's why

$$|q(\mathrm{x}+\tau) - q(\mathrm{x})| =$$

$$= \left|\sin(x+\tau) + \sin\sqrt{2}\,(x+\tau) - \sin x - \sin\sqrt{2}x\right| \le$$

$$\le |\sin(x+\tau) - \sin x| + \left|\sin\sqrt{2}(x+\tau) - \sin\sqrt{2}x\right| =$$

$$= |\sin(x+\tau-2\pi\mathrm{n}_1) - \sin x| + \left|\sin(\sqrt{2}x + \sqrt{2}\tau - 2\pi\mathrm{n}_2) - \sin\sqrt{2}x\right| = \left|2\cos\tfrac{1}{2}(2x+\tau-2\pi\mathrm{n}_1)\sin\tfrac{1}{2}(\tau - 2\pi\mathrm{n}_1)\right| + \left|2\cos\tfrac{1}{2}(2\sqrt{2}x + \sqrt{2}\tau - 2\pi\mathrm{n}_1)\sin\tfrac{1}{2}(\sqrt{2}\tau - 2\pi\mathrm{n}_1)\right| \le 4\sin\tfrac{\delta}{2}.$$

And since 8 can be chosen as small as desired, the difference $q(x+\tau) - \mathrm{q}(x)$, with corresponding m, will be arbitrarily small in absolute value.

In connection with this example, we come to the main thing for the whole theory the concept of displacement or almost period.

Definition 2 [5]. Number τ called displacement {almost period)

functions $f(x)$ corresponding to the number ε (ε -displacement, ε -pochgpi period), if inequality holds

$$\sup_{-\infty<\mathrm{x}<\infty} |f(x+\tau) - f(x)| < \varepsilon.$$

Note that if τ is an ε -displacement, then τ is also an ε -displacement. If τ_1 is an ε_1-displacement and τ_2 is an ε_1 -displacement, then the numbers $\tau_1 \pm \tau_2$ are $\varepsilon_1 \pm \varepsilon_2$ -displacement. The last statement follows from the inequality

$$\sup_{-\infty<\mathrm{x}<\infty} |f(x+\tau_1 \pm \tau_2) - f(x)| \le$$
$$\le \sup_{-\infty<\mathrm{x}<\infty} |f(x+\tau_1 \pm \tau_2) - f(x+\tau_1)| +$$
$$\sup_{-\infty<\mathrm{x}<\infty} |f(x+\tau_1 \pm \tau_2) - f(x+\tau_2)| < \varepsilon_1 + \varepsilon_2.$$

Let $q(x)$ be a periodic function and τ its period. Then oh it is clear that τ will also be almost a period for q(x), corresponding to any $\varepsilon > 0$.

If the function $f(x)$ is uniformly continuous over the entire real axis, then for any $\varepsilon > 0$ there are always sufficiently small displacements, however, it is clear that these shifts are not of particular interest.

It is natural to demand the existence for every $\varepsilon > 0$ arbitrarily large displacements. But if we limit ourselves only to this requirement, then as Bohr showed (in the appendix to the first main memoir), we do not we obtain a linear class of functions; in other words, the sum of two functions, each of which has, for any $\varepsilon > 0$, arbitrarily large offsets will not always satisfy the same condition.

Therefore, the requirements imposed on offsets should be strengthened.

For this purpose, we introduce the concept of a relatively dense set.

Definition 3. The set E of real numbers is called

relatively dense t if there exists a number $l > 0$ such that in each interval valid axes length l $(a < x < a + l)$ there will be at least one plural number E.

For example, the numbers of the arithmetic progression $np(n = 0, \pm 1, \pm 2, \dots)$ form a relatively dense set just like numbers of the form $\pm\sqrt{n}$ (n is an integer, positive). On the contrary, numbers of the form $\pm n^2$ are not form a relatively dense set.

Definition 4 [3, 5]. A function $f(x)$ continuous on the entire real axis is called uniform almost periodic if for each $\varepsilon > 0$ one can specify positive numbers $l = l(\varepsilon)$ such that in each interval of length l there is at least one number τ, for which

$$|f(x+\tau) - f(x)| < \varepsilon \quad (x \in R).$$

The space of such functions with norm

$$\|f(x)\|_B = \sup_{x\in R}|f(x)|$$

denote by B and write the Fourier series of the function $f(x) \in B$ in the form

$$f(x) \sim \sum_{k=-\infty}^{\infty} A_k \exp(i\lambda_k x),$$

$$A_k = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(x)\exp(-i\lambda_k x)\,dx,$$

where the numbers $\{\lambda_k\}$ are Fourier exponents that have a single limit point at zero, that is

$$\lambda_k > 0\ (k > 0),\ \lambda_{-k} = -\lambda_k,\ |\lambda_k| < |\lambda_{k-1}|,\ (k = 1, 2, \ldots), \lim_{k\to\infty} |\lambda_k| = 0. \quad (2)$$

In this paper we will indicate some sufficient conditions for the convergence of the series

$$\sum_{k=-\infty}^{\infty} |A_k|^{\beta}\, |k|^{\gamma} (\gamma > 0, \beta > 0). \quad (3)$$

For the function $f(x) \in B$, consider the integral representation

$$F(x) = \theta \int_0^{\infty} e^{-t\theta} f(x-t)dt\ (\theta > 0).$$

From the theorem on the indefinite integral of uniform almost periodic functions it follows that $F(x) \in B$ (see [5], p. 29). For $\theta > 0$, we introduce into consideration the quantity $\Omega(f;\theta) = \theta \left\{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left| \int_0^{\infty} e^{-\theta t} f(x-t)dt \right|^p dx\right\}^{1/p}$.

Note that the value $\Omega(f;\theta)$ in the case when the spectrum of the function $f(x) \in B$ satisfies conditions (3) is an analogue of the modulus of continuity.

From the definition of almost periodic functions it immediately follows:

Theorem 1. The almost periodic function $f(x,y)$ is bounded, that is, there is a number $C = C(f)$ such that for all $x\ (-\infty < x < \infty)$

$$|f(x)| \in C.$$

Proof. Let us first determine, for example for $\varepsilon = 1$, the length $L = L(\varepsilon)$=L(1). Function f(x) as a continuous function, bounded in closed intervals $0 \le x \le l(1)$, let's say $|f(x)| \in C$. We will prove that then at each point x_0 the inequalities $f(x_0) \le c + 1$ are satisfied. Indeed, for any x_0 there exists $\tau = \tau(1)$ such that $0 < x_0 + \tau > l(1)$. Next we have:

$$|f(x_0)| = |f(x_0) - f(x_0+\tau) + f(x_0+\tau)| \le |f(x_0+\tau)| + |f(x_0) - f(x_0+\tau)| < 1 + c = C.$$

Theorem 1 is proven.

In order for the indefinite integral of a periodic function f(x,y) to also be a periodic function, it is necessary and sufficient that the Fourier series of the function f(x,y) does not contain a free term. However, for the function of an indefinite integral to be almost periodic, in general, the absence of a free term in the Fourier series of almost periodic functions is not sufficient. If the indefinite integral of uniform almost-periodic functions is a uniform almost-periodic function, then by Theorem 1. It is necessarily bounded.

Theorem 2. If the indefinite integral of uniform almost-periodic functions is bounded, then it is also a network of uniform almost-periodic functions

Proof. We can obviously limit ourselves to only real functions. By condition function

$$P(x) = \int_0^x f(x,y)dx + C,$$

limited. Let us denote its upper bound by G. We will show that for every $\varepsilon > 0$ there is a positive number $\varepsilon_1 = \varepsilon_1[\varepsilon, f(x)]$ such that every ε_1 is an almost-period of the function f(x,y) there is \varepsilon – the almost-period of the function $f(x)$. For this purpose, we choose two fixed numbers $x_1$and x_2 so that the inequalities are satisfied

$$\mathrm{F}(x_1) < g + \frac{\varepsilon}{6},\ \mathrm{F}(x_2) > G + \frac{\varepsilon}{6}$$

Set $|x_1 - x_2| = d$, $|y_1 - y_2| = d$ and $min\ (x_1, x_2) = \xi$.

Let us assume that in each interval of length $l_0 = l\left(\frac{\epsilon}{6d}\right)$, there is at least one $\frac{\epsilon}{6d}$- almost-period of the function $f(t)$. Due to the fact th*at $f(x) \in B$, there is such a number l_0. Before proving P(x), we will show that in each intervals $(\alpha, \alpha + L_0)$ $L_0 = l_0 + d$, there are values u_1 and u_2 such that*

$$P(u_1) < g + \frac{\varepsilon}{2}, \quad (4)$$

$$P(u_2) < G - \frac{\varepsilon}{2} \quad (5)$$

In fact, we can choose the almost-period $\tau = \tau\left(\frac{\varepsilon}{6d}\right)$ so that the numbers $\xi = \tau$ lie in the intervals $(\alpha, \alpha + l_0)$. Then both numbers $u_1 = x_1 + \tau, u_2 = x_2 + \tau$ will probably lie in larger intervals $(\alpha, \alpha + L_0)$ and we will have:

$$P(u_2) - P(u_1) = P(x_2) - P(x_1) + + \int_{u_1}^{u_1} f(\mathrm{z})d\mathrm{z} - \int_{x_1}^{x_2} f(\mathrm{z})d\mathrm{z} = = P(x_2) - P(x_1) + \int_{x_1}^{x_2} [f(z+\tau) - f(\mathrm{z})]dz \ge P(x_2) - P(x_1) - d\frac{\varepsilon}{6d} > G - g - \frac{2\varepsilon}{6} - \frac{\varepsilon}{6} = G - g - \frac{\varepsilon}{2}$$

But the inequalities $P(x_2) - P(x_1) > G - g - \frac{\varepsilon}{2}$ in the sense of the numbers G and g is possible only if inequalities (4) and (5) are satisfied.

We will now show that the number $\varepsilon_1 = \frac{\varepsilon}{2L_0}$ has the desired property, that is, that every $\varepsilon_1$almost-period $f(x)$ is a ε almost-period $P(x)$

Let us show separately the validity of each inequality:

$$P(x+\tau) - P(x) > -\varepsilon, \quad (6)$$

$$P(x+\tau) - P(x) < \varepsilon. \quad (7)$$

To prove inequality (6), we choose in the intervals $(x, x + L_0)$ (x is an arbitrary real number) values u_1 such that $P(u_1) < g + \frac{\varepsilon}{2}, P(v_1) < g + \frac{\varepsilon}{2}$. Then

$$\mathrm{P}(x+\tau)-P(x)=$$
$$=P(u_1+\tau)-P(u_1)+\int_x^{x+\tau} f(\mathrm{z})d\mathrm{z}-\int_{u_1}^{u_1+\tau} f(\mathrm{z})d\mathrm{z}=$$
$$P(u_1+\tau)-P(u_1)+\int_x^{u_1} f(\mathrm{z})d\mathrm{z}-z\geq g-\left(g+\frac{\varepsilon}{2}\right)-$$
$$\left|\int_x^{u_1}[f(z+\tau)-f(\mathrm{z})]d\mathrm{z}\right|>-\frac{\varepsilon}{2}-L_0\frac{\varepsilon}{2L_0}=-\varepsilon.$$

To prove inequality (7), we select in the intervals $(x, x+L_0)$ a point u_2, in which $P(u_2)>G-\frac{\varepsilon}{2}$, $P(v_2)>G-\frac{\varepsilon}{2}$. Then

$$P(x+\tau)-P(x)=$$
$$=P(u_2+\tau)-P(u_2)+\int_x^{u_2} f(\mathrm{z})d\mathrm{z}-\int_{x+\tau}^{u_2+\tau} f(\mathrm{z})d\mathrm{z}<$$

$$<G-\left(G-\frac{\varepsilon}{2}\right)+\int_{x,y}^{u_2} e^{-\theta z}|f(z+\tau)-f(z)|\,dz<\frac{\varepsilon}{2}+L_0\cdot\frac{\varepsilon}{2L_0}=\varepsilon.$$

Theorem 2 is proven.

Theorem 3. If for a function $f(x)\in B$, the spectrum of which satisfies conditions (2), the series converges

$$\sum_{\nu=0}^{\infty} 2^{\nu\left(\gamma+\frac{q-\beta}{q}\right)}\Omega(f;\lambda_{2^\nu}), \tag{8}$$

where $1<p\leq 2, \frac{1}{p}+\frac{1}{q}=1, 0<\beta<q, \gamma>0$, then series (3) converges.

Proof. Let us show that for the function F(x) the Fourier series has the form

$$\sum_{k=-\infty}^{\infty}\frac{\theta A_k}{\theta+i\lambda_k}e^{i\lambda_k x}.$$

Really,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}F(x)\,e^{-i\lambda_k x}dx=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\theta\int_0^\infty e^{-\theta t}f(x-t)dte^{-i\lambda_k x}dx=$$

$$=\theta\int_0^\infty\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x-t)\,e^{-i\lambda_k x}dx\right]e^{-\theta t}dt=\theta A_k\int_0^\infty e^{-(\theta+i\lambda_k)t}dt=\frac{\theta A_k}{\theta+i\lambda_k}.$$

By virtue of the Hausdorff-Young inequality, the proof of which is also true for functions $f(x)\in B$, we have

$$\Omega(f;\theta)=\theta\left\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|\int_0^\infty e^{-\theta t}f(x-t)dt\right|^p dx\right\}^{1/p}\geq\left\{\theta^q\sum_{k=-\infty}^{\infty}\left|\frac{A_k}{\theta+i\lambda_k}\right|^q\right\}^{1/q}\ (1<p\leq 2). \tag{9}$$

Substituting $\theta=\lambda_{2^{\nu-1}}$ into (9), we obtain

$$2^{-\frac{q}{2}}\sum_{k=2^{\nu-1}+1}^{2^\nu}|A_k|^q<\Omega^q(f;\lambda_{2^{\nu-1}}). \tag{10}$$

Using Hölder's inequality and (10), we have

$$\sum_{k=2^{\nu-1}+1}^{2^\nu}|A_k|^\beta|k|^\gamma\leq\left\{\sum_{k=2^{\nu-1}+1}^{2^\nu}|A_k|^q\right\}^{\frac{\beta}{q}}\left\{\sum_{k=2^{\nu-1}+1}^{2^\nu}k^{\frac{\gamma q}{q-\beta}}\right\}^{1-\frac{\beta}{q}}\leq$$

$$\leq 2^{\frac{\beta}{2}}\Omega^\beta(f;\lambda_{2^{\nu-1}})\cdot\left(2^{\frac{\nu\gamma q}{q-\beta}}\cdot 2^{\nu-1}\right)^{\frac{q-\beta}{q}}=2^{\frac{\beta}{2}}\Omega^\beta(f;\lambda_{2^{\nu-1}})\cdot 2^{\nu\gamma+\frac{(\nu-1)(q-\beta)}{q}}=2^{\frac{\beta}{2}+\gamma}2^{(\nu-1)\left(\gamma+\frac{q-\beta}{q}\right)}\Omega^\beta(f;\lambda_{2^{\nu-1}}).$$

It follows that

$$\sum_{k=2^{\nu-1}+1}^{2^\nu}|A_k|^\beta|k|^\gamma\leq C2^{(\nu-1)\left(\gamma+\frac{q-\beta}{q}\right)}\Omega^\beta(f;\lambda_{2^{\nu-1}}), \tag{11}$$

where the constant C depends on β and γ. Summing inequality (11) over ν, we obtain

$$\sum_{k=2}^{\infty}|A_k|^\beta|k|^\gamma<C\sum_{\nu=0}^{\infty}2^{\nu\left(\gamma+\frac{q-\beta}{q}\right)}\Omega^\beta(f;\lambda_{2^{\nu-1}}). \tag{12}$$

Since $\lambda_{-k}=-\lambda_k$, then taking $\theta=|\lambda_{-2^{\nu-1}}|$ in inequality (9), we will have

$$2^{-\frac{q}{2}}\sum_{k=-2^\nu}^{-(2^{\nu-1}+1)}|A_k|^q\leq\Omega^q(f;|\lambda_{-2^{\nu-1}}|)=\Omega^q(f;\lambda_{2^{\nu-1}}).$$

Similarly, as was established (12), it can be obtained that

$$\sum_{k=-2}^{\infty}|A_k|^\beta k^\gamma<C\sum_{\nu=0}^{\infty}2^{\nu\left(\gamma+\frac{q-\beta}{q}\right)}\Omega^\beta(f;\lambda_{2^\nu}). \tag{13}$$

The statement of Theorem 1 follows from inequalities (12) and (13).

To formulate the following result, we introduce the following notation:

$$G_n = \{k\colon 2^{-n-1} \leq \lambda_k < 2^{-n}\};$$

$$G_{-n} = \{k\colon -2^{-n-1} \leq \lambda_k < -2^{-n}\};$$

$$M_n = \max_{k \in G_n} |k|\,;\ \mu(a) = \sum_{\lambda_k \geq a} 1.$$

Theorem 4. *Let* $f(x) \in B$ *and its spectrum* $\Lambda\{\lambda_k\}_{k=-\infty}^{\infty}$ *satisfy conditions (1). If*

$$\sum_{n=1}^{\infty} M_n^{\gamma} \{\mu(2^{-n-1}) - \mu(2^{-n})\}^{1-\frac{\beta}{q}} \Omega^{\beta}\left(f; \frac{1}{2^n}\right) < \infty,$$

then series (3) converges.

Proof. Substituting $\theta = \frac{1}{2^n}$ into inequality (10), we get

$$\sum_{k \in G_n} |A_k|^q < 2^{\frac{q}{2}} \Omega^{q}\left(f; \frac{1}{2^n}\right).$$

From here, using Hölder's inequality, we have

$$\sum_{k \in G_n} |A_k|^{\beta}\, k^{\gamma} \leq M_n^{\gamma} [\mu(2^{-n-1}) - \mu(2^{-n})]^{1-\frac{\beta}{q}} \cdot 2^{\frac{\beta}{2}} \Omega^{\beta}\left(f; \frac{1}{2^n}\right).$$

Summing the last over n, we find

$$\sum_{n=1}^{\infty} \sum_{k \in G_n} |A_k|^{\beta}\, k^{\gamma} \leq 2^{\frac{\beta}{2}} \sum_{n=1}^{\infty} M_n^{\gamma} [\mu(2^{-n-1}) - \mu(2^{-n})]^{1-\frac{\beta}{q}} \Omega^{\beta}\left(f; \frac{1}{2^n}\right). \tag{14}$$

If we take into account $G_{-n} = -G_n$ (this follows from the equality $\lambda_{-k} = -\lambda_k$) and the convergence of elements of the sets G_{-n} и and G_n, then, similar to inequality (14), we arrive at the following inequality

$$\sum_{n=1}^{\infty} \sum_{k \in B_{-n}} |A_k|^{\beta}\, |k|^{\gamma} \leq 2^{\frac{\beta}{2}} \sum_{n=1}^{\infty} M_n^{\gamma} [\mu(2^{-n-1}) - \mu(2^{-n})]^{1-\frac{\beta}{q}} \Omega^{\beta}\left(f; \frac{1}{2^n}\right).$$

The last inequality and (14) imply the statement of Theorem 2.

Let the function $f(x) \in B$ for some number α $(0 < \alpha \leq 1)$ satisfy the condition

$$\left|\int_0^u f(x-t)dt\,\right| \leq C|u|^{1-\alpha}. \tag{15}$$

Let us show that the following relation is valid

$$\Omega(f;\theta) \leq I(t)\theta^{\alpha}, I(t) = C\int_0^{\infty} e^{-t} t^{1-\alpha} dt, \tag{16}$$

where C is a constant. Indeed, integrating by parts the inner integral on the left side of (16), we obtain

$$\left|\int_0^{\infty} e^{-\theta t} f(x-t)dt\right| \leq C\theta^{\alpha-1} \int_0^{\infty} e^{-t} t^{1-\alpha} dt = I(t)\theta^{\alpha-1}$$

that is, condition (15) implies relation (16). Consequently, the following corollaries follow from Theorems 3 and 4.

Corollary 1. *Let the function* $f(x) \in B$ *satisfy condition (10) and*

$$\sum_{n=1}^{\infty} n^{\gamma-\frac{\beta}{q}} \lambda_n^{\alpha\beta} < \infty, \tag{17}$$

then series (3) converges.

In fact, since condition (15) implies inequality (16), then

$$\sum_{\nu=1}^{\infty} 2^{\nu\left(\gamma+\frac{q-\beta}{q}\right)} \Omega^{\beta}(f; \lambda_{2^{\nu}}) \leq I(t) \sum_{\nu=1}^{\infty} 2^{\nu\left(\gamma+\frac{q-\beta}{q}\right)} \lambda_{2^{\nu}}^{\alpha\beta}.$$

Due to the monotonicity of the sequence $\Lambda\{\lambda_n\}_{n=1}^{\infty}$, the convergence of the series on the right side of the last inequality is equivalent to the convergence of the series (17).

Corollary 2. If the function $f(x) \in B$ satisfies conditions (15) and

$$\sum_{n=1}^{\infty} G_n^{\gamma} [\mu(2^{-n-1}) - \mu(2^{-n})]^{1-\frac{\beta}{q}} 2^{-n\alpha} < \infty,$$

then series (3) converges.

Note that if in inequality (16) we assume $\gamma = 0, \beta = 1, p = 2, \lambda\_n = O(\frac{1}{n})$, then for $\gamma > \frac{1}{2}$ the Fourier series converges absolutely. This result was established by N. P.

Kuptsov [5]. And for the Besikovich function, similar results were obtained in the works of Yu. Kh. Khasanov (see, for example, [8, 16]), but instead of the value $\Omega(f;h)$, an averaging module of order k was used

$$W_k(f;H)_{B_p} = \sup_{T\geq H}\sup_{x\in R}\left|f_{T^k}(x)\right|_{B_p} (\mathrm{H}>0, k\in N),$$

Where

$$f_{T^k}(x) = \frac{1}{(2T)^k}\int_{x-T}^{x+T} dt_1 \int_{t_1-T}^{t_1+T} dt_2 \dots \int_{t_{k-2}-T}^{t_{k-2}+T} dt_{k-1} \int_{t_{k-1}-T}^{t_{k-1}+T} f(t_k)\, dt_k.$$

3. Materials and Methods

The work uses methods of function theory and functional analysis of an approximative nature, methods for solving problems of harmonic analysis for functions, the theory of Fourier series, the theory of summation of Fourier series and methods of approximating functions by trigonometric polynomials.

4. Results

The results of the work are new, obtained by the author independently and are as follows:

Sufficient conditions have been found for the absolute convergence of Fourier series of uniform almost periodic functions when: a) their spectrum has a unique limit point at infinity; b) their spectrum has a single limit point at zero, while as a structural characteristic of functions, a value constructed on the basis of the Laplace transform is used.

5. Discussion

The work is both theoretical and practical in nature. The results of the work can be applied in the theory of Fourier series and special sections of function theory.

6. Conclusions

The main scientific results of the work are as follows:

Sufficient conditions have been found for the absolute convergence of Fourier series of uniform almost periodic functions when: a) their spectrum has a unique limit point at infinity; b) their spectrum has a single limit point at zero, while as a structural characteristic of functions, a value constructed on the basis of the Laplace transform is used.

Conflicts of Interest

The authors declare no conflicts of interest.

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