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Some Fixed Point Theorems on b_2 - Metric Spaces

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Abstract: In this study, we generalize both b-metric spaces and 2-metric spaces into a new class of generalized metric spaces that we call b_2-metric spaces. Then, under various contractive circumstances in partially ordered spaces, we demonstrate a few fixed point theorems in b_2-metric space. Many Mathematician gave the concept of b_2 -metric spaces as a generalization of 2-metric space. The purpose of this research article to established some results of 2-metric space proved by the Arun Garg et al. in b_2 -metric spaces and prove new results.

Keywords: Fixed Point, b - Metric Space, 2-Metric Space, Partial Order Set, Generalized Contractive Mappings

1. Introduction

The idea of metric spaces in functional analysis was initially introduced by Maurice Frechet in 1906 [29]. Mathematicians have since developed the idea of metric spaces in a variety of ways.

Czerwik and many other writers investigated, introduced, and proved various fixed point solutions for single and multi valued mappings in 1993 [1, 2]. Czerwik also studied, introduced, and proved the idea of a b-metric space.

On the other hand, Gähler offered the idea of a 2-metric in [3], using the encouraging example of a triangle's area in R3. For mappings in these spaces, multiple fixed point results were also attained. It is important to keep in mind that 2-metric spaces are not topologically equal to metric spaces, unlike many other recent generalizations of metric spaces, and there is no direct connection between the conclusions produced in 2-metric and in metric spaces.

Different Mathematician studied the various types of mappings in b-Metric Space and 2-Metric Spaces [4-26].

As a generalization of both 2-metric and b-metric spaces, Zead Mustafa et. al. [27] offer the idea of b_2-metric spaces in their study. Then, in partially ordered b_2-metric spaces, he established a few fixed point theorems under various contractive circumstances.

We expand the findings of the b-complete b-metric space in to b_2-metric spaces in this study.

2. Mathematical Preliminaries

The definitions provided by Zead Mustafa et al. [27] are as follows:

Definition 1: Let χ be a nonempty set, $s \geq 1$ be a real number and let $\delta : \chi^3 \to \Re$ be a map satisfying the following conditions:

1. For every pair of distinct points $x, y \in \chi$ there exists a point $z \in \chi$ such that $\delta(x, y, z) \neq 0$
2. If two of three points x, y, z are same then $\delta(x, y, z) = 0$
3. The symmetry:

$$\delta(x,y,z) = \delta(x,z,y) = \delta(y,x,z) = \delta(y,z,x) = \delta(z,x,y) = \delta(z,y,x) \text{ for all } x, y, z \in \chi.$$

4. The rectangle inequality: $\delta(x, y, z) \leq s[\delta(x, y, t) + \delta(y, z, t) + \delta(z, x, t)]$ for all $x, y, z, t \in \chi$.

Definition 2: Let $\{x_n\}$ be a sequence in a b_2-metric space (χ, δ). Then

1. $\{x_n\}$ is said to be b_2-convergent to $x \in \chi$, written as $\lim_{n\to\infty} x_n = x$ if for all $a \in \chi$, $\lim_{n\to\infty} \delta(x_n, x, a) = 0$.
2. $\{x_n\}$ is said to be a b_2-Cauchy sequence in χ if for all $a \in \chi$, $\lim_{n\to\infty} \delta(x_n, x_m, a) = 0$.
3. (χ, δ) is said to be b_2-complete if every b_2-Cauchy sequence is a b_2-Convergent sequence.

Some simple b_2-metric space examples are provided below [27]:

Example 1: Let $\phi = [0, \infty)$ and $\delta(x, y, z) = [xy, yz, zx]^p$ *if* $x \neq y \neq z \neq x$, and otherwise

$\delta(x, y, z) = 0$, where $p \geq 1$ is a real number. Evidently, from convexity of function

$f(x) = x^p$ *for* $x \geq o$, then by Jensen inequality, we have

$$(a+b+c)^p \leq 3^{p-1}(a^p + b^p + c^p).$$

So, one can obtain the result that (χ, δ) b_2-metric space with $s \leq 3^{p-1}$.

Example 2: Let a mapping $\delta : \Re^3 \to [0, +\infty)$ be defined by

$$\delta(x, y, z) = \min\{|x-y|, |y-z|, |z-x|\}$$

Then δ is a 2-metric on $\Re$, *i. e.*, the following inequality holds:

$$\delta(x, y, z) \leq \delta(x, y, t) + \delta(y, z, t) + \delta(z, x, t)$$

for arbitrary real numbers *x*, *y*, *z*, *t*. Using convexity of the function

$f(x) = x^p$ on $[0, +\infty)$ for $p \geq 1$, we obtain that

$\delta_p = \min\{|x-y|, |y-z|, |z-x|\}^p$ is a b_2- *metric on* $\Re$ *with* $s < 3^{p-1}$.

Proposition 1: Let (χ, δ) and $(\chi^\gamma, \delta^\gamma)$ be two b_2-metric spaces. Then a

Mapping $f : \chi \to \chi^\gamma$ is b_2-continuous at a point $x \in \chi$ if and only if it is b_2-sequentially continuous at x; that is, whenever $\{x_n\}$ is b_2-

Convergent to x, $\{fx_n\}$ is b_2-convergent to $f(x)$.

Lemma 1 [27]: *Let* (χ, δ) *be a* b_2*-metric space and suppose that* $\{x_n\}$ *and*

$\{y_n\}$ *are* b_2*-convergent to x and y, respectively. Then we have*

$$\frac{1}{s^2}\delta(x, y, a) \leq \underset{n\to\infty}{Lim}\inf \delta(x_n, y_n, a) \leq \underset{n\to\infty}{Lim}\sup \delta(x_n, y_n, a) \leq s^2 \delta(x, y, a)$$

for all "a" in χ. *In particular, if* $y_n = y$ *is constant, then*

$$\frac{1}{s}\delta(x, y, a) \leq \underset{n\to\infty}{Lim}\inf \delta(x_n, y_n, a) \leq \underset{n\to\infty}{Lim}\sup \delta(x_n, y_n, a) \leq s\delta(x, y, a)$$

Proof: It is simple to observe that using the rectangle inequality in the provided b_2-metric space

$$\delta(x, y, a) = \delta(x, a, y) \leq s\delta(x, a, x_n) + s\delta(a, y, x_n) + s\delta(y, x, x_n)$$

$$\leq s\delta(x, a, x_n) + s^2[\delta(a, y, y_n) + \delta(y, x_n, y_n) + \delta(x_n, a, y_n)] + s\delta(y, x_n, y_n)$$

And

$$\delta(x_n, y_n, a) = \delta(x_n, a, y_n) \leq s\delta(x_n, a, x) + s\delta(a, y_n, x) + s\delta(y_n, x, x_n)$$

$$\leq s\delta(x_n, a, x) + s^2[\delta(a, y_n, y) + \delta(y_n, x, y) + \delta(x, a, y)] + s\delta(y_n, x, x_n)$$

We get the desired outcome by using $n \to \infty$ n as the upper limit in the second inequality and $n \to \infty$ as the lower limit in the first inequality.

If $y_n = y$, then

$$\delta(x, y, a) \leq s\delta(x, y, x_n) + s\delta(y, a, x_n) + s\delta(a, x, x_n)$$

And

$$\delta(x_n, y, a) \leq s\delta(x_n, y, x) + s\delta(y, a, x) + s\delta(a, x_n, x)$$

Main Results:

We begin by demonstrating a lemma that states the sequence $\{x_n\}$ is a b_2-Cauchy sequence.

Lemma 2: Let (χ, ∂) be a b_2- metric space with coefficient $s \geq 1$ and $\Gamma : \chi \to \chi$ be a mapping.

Suppose that $\{x_n\}$ is a sequence in χ induced by $x_{n+1} = \Gamma x_n$ such that

$$\partial(x_n, x_{n+1}, a) \leq \alpha\partial(x_{n-1}, x_n, a) \tag{1}$$

For all $n \in N$, where $\alpha \in [0,1)$ is a constant. Then $\{x_n\}$ is a b_2- Cauchy sequence.

Proof: Suppose $x_0 \in \chi$ and $x_{n+1} = \Gamma x_n$ for all $n \in N$. For the lemma's proof, three separate cases are taken into account.

Case I: Let $\alpha \in [0, \frac{1}{s})$. By (1), we have

$$\partial(x_n, x_{n+1}, a) \le \alpha\partial(x_{n-1}, x_n, a)$$

$$\le \alpha^2\partial(x_{n-2}, x_{n-1}, a)$$

$$\le \alpha^3\partial(x_{n-3}, x_{n-2}, a)$$

$$\le \alpha^n\partial(x_0, x_1, a)$$

Thus, for any $n \ge m$ and $n, m \in N$, we have

$$\partial(x_m, x_n, a) \le s[\partial(x_m, x_{m+1}, a) + \partial(x_{m+1}, x_n, a)]$$

$$\le s\partial(x_m, x_{m+1}, a) + s^2[\partial(x_{m+1}, x_{m+2}, a) + \partial(x_{m+2}, x_n, a)]$$

$$\le s\partial(x_m, x_{m+1}, a) + s^2\partial(x_{m+1}, x_{m+2}, a) + s^3[\partial(x_{m+2}, x_{m+3}, a) + \partial(x_{m+3}, x_n, a)]$$

$$\le s\partial(x_m, x_{m+1}, a) + s^2\partial(x_{m+1}, x_{m+2}, a) + s^3\partial(x_{m+2}, x_{m+3}, a) + s^4\partial(x_{m+3}, x_{m+4}, a) + \\ ------ + s^{n-m-1}\partial(x_{n-2}, x_{n-1}, a) + s^{n-m-1}\partial(x_{n-1}, x_n, a)$$

$$\le s\alpha^m\partial(x_0, x_1, a) + s^2\alpha^{m+1}\partial(x_0, x_1, a) + s^3\alpha^{m+2}\partial(x_0, x_1, a) + s^4\alpha^{m+3}\partial(x_0, x_1, a) + \\ .... + s^{n-m-1}\alpha^{n-2}\partial(x_0, x_1, a) + s^{n-m-1}\alpha^{n-1}\partial(x_0, x_1, a)$$

$$\le s\alpha^m[1 + s\alpha + s^2\alpha^2 + s^3\alpha^3 + s^4\alpha^4 + ... + s^{n-m-1}\alpha^{n-m-2} + s^{n-m-1}\alpha^{n-m-1}]\partial(x_0, x_1, a)$$

$$\le s\alpha^m[\sum_{i=0}^{\infty}(s\alpha)^i]\partial(x_0, x_1, a)$$

$= \frac{s\alpha^m}{1 - s\alpha}\partial(x_0, x_1, a)$, as $m \to \infty$, which implies that $\{x_n\}$ is a b_2- Cauchy sequence.

In other words $\{\Gamma^n x_0\}$ is a b_2- Cauchy sequence.

Case II: Now, let $\alpha \in [\frac{1}{s}, 1), (s > 1)$. In this case, we have $\alpha^n \to 0$ as $n \to \infty$, so there is $n_0 \in N$, such that $\alpha^{n_0} < s$. Thus, by case I, we claim that $\{(\Gamma^{n_0})^n x_0\}_{n=0}^{\infty} := \{x_{n_0}, x_{n_0+1}, x_{n_0+2}, ..., x_{n_0+n}.......\}$ is a b_2- Cauchy sequence. Then

$\{x_n\}_{n=0}^{\infty} := \{x_0, x_1, x_2, ..., x_{n_0-1}, ...\} \cup \{x_{n_0}, x_{n_0+1}, x_{n_0+2}, ..., x_{n_0+n}.......\}$

is a b_2- Cauchy sequence in X.

Case III: Let s=1, then the proof of lemma is similar to case I.

Now we prove the theorems of Arun Garg et. al [28] in b_2-metric spaces:

Theorem 3: Let $(\chi, \le)$ be a partially ordered set and suppose that there exist a b_2- metric ∂ on χ such that (χ, ∂) is a b_2- complete metric space with coefficient $s \ge 1$ and $\Gamma : \chi \to \chi$ be a mapping such that

$$s\partial(\Gamma x, \Gamma y, a) \le \alpha_1\partial(x, y, a) + \alpha_2\frac{\partial(x, \Gamma x, a)\partial(y, \Gamma y, a)}{1 + \partial(x, y, a)} + \alpha_3\frac{\partial(x, \Gamma y, a)\partial(y, \Gamma x, a)}{1 + \partial(x, y, a)} + \\ \alpha_4\frac{\partial(x, \Gamma x, a)\partial(x, \Gamma y, a)}{1 + \partial(x, y, a)} + \alpha_5\frac{\partial(y, \Gamma x, a)\partial(y, \Gamma y, a)}{1 + \partial(x, y, a)} \tag{2}$$

Where α_1, α_2, α_3, α_4 and α_5 are positive constant with $(\alpha_1 + \alpha_2 + \alpha_3 + s\alpha_4 + s\alpha_5) < 1$. Then Γ has a unique fixed point in χ. Moreover, for any $x \in \chi$, the iterative sequence $\{\Gamma^n x\}$ ($n \in N$) b_2- converges to fixed point.

Proof: Assuming $x_0 \in \chi$ such, we create an iterative sequence $\{x_n\}$ by $x_{n+1} = \Gamma x_n$ ($n \in N$).

If there exist $n_0 \in N$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} = x_{n_0+1} = \Gamma x_{n_0}$, I e. x_{n_0} is a fixed point of Γ.

Without losing generality, let's move on, suppose $x_n \ne x_{n+1}$ for all ($n \in N$), then by (2)

$$s\partial(x_n,x_{n+1},a)=s\partial(\Gamma x_{n-1},\Gamma x_n,a)$$
$$\le \alpha_1\partial(x_{n-1},x_n,a)+\alpha_2\frac{\partial(x_{n-1},\Gamma x_{n-1},a)\partial(x_n,\Gamma x_n,a)}{1+\partial(x_{n-1},x_n,a)}+\alpha_3\frac{\partial(x_{n-1},\Gamma x_n,a)\partial(x_n,\Gamma x_{n-1},a)}{1+\partial(x_{n-1},x_n,a)}+$$
$$\alpha_4\frac{\partial(x_{n-1},\Gamma x_{n-1},a)\partial(x_{n-1},\Gamma x_{n-1},a)}{1+\partial(x_{n-1},x_n,a)}+\alpha_5\frac{\partial(x_n,\Gamma x_{n-1},a)\partial(x_n,\Gamma x_n,a)}{1+\partial(x_{n-1},x_n,a)}$$

$$\le \alpha_1\partial(x_{n-1},x_n,a)+\alpha_2\frac{\partial(x_{n-1},x_n,a)\partial(x_n,x_{n+1},a)}{1+\partial(x_{n-1},x_n,a)}+\alpha_3\frac{\partial(x_{n-1},x_{n+1},a)\partial(x_n,x_n,a)}{1+\partial(x_{n-1},x_n,a)}+$$
$$\alpha_4\frac{\partial(x_{n-1},x_n,a)\partial(x_{n-1},x_n,a)}{1+\partial(x_{n-1},x_n,a)}+\alpha_5\frac{\partial(x_n,x_n,a)\partial(x_n,x_{n+1},a)}{1+\partial(x_{n-1},x_n,a)}$$

$$\le \alpha_1\partial(x_{n-1},x_n,a)+\alpha_2\partial(x_n,x_{n+1},a)+\alpha_4 s[\partial(x_{n-1},x_n,a)+\partial(x_n,x_{n+1},a)]$$

$$s\partial(x_n,x_{n+1},a)\le(\alpha_1+s\alpha_4)\partial(x_{n-1},x_n,a)+(\alpha_2+s\alpha_4)\partial(x_n,x_{n+1},a)$$

$$(s-\alpha_2-s\alpha_4)\partial(x_n,x_{n+1},a)\le(\alpha_1+s\alpha_4)(x_{n-1},x_n,a) \tag{3}$$

Again,

$$s\partial(x_n,x_{n+1},a)=s\partial(\Gamma x_{n-1},\Gamma x_n,a)$$
$$\le \alpha_1\partial(x_{n-1},x_n,a)+\alpha_2\frac{\partial(x_{n-1},\Gamma x_{n-1},a)\partial(x_n,\Gamma x_n,a)}{1+\partial(x_{n-1},x_n,a)}+\alpha_3\frac{\partial(x_n,\Gamma x_{n-1},a)\partial(x_{n-1},\Gamma x_n,a)}{1+\partial(x_{n-1},x_n,a)}+$$
$$\alpha_4\frac{\partial(x_n,\Gamma x_n,a)\partial(x_n,\Gamma x_{n-1},a)}{1+\partial(x_{n-1},x_n,a)}+\alpha_5\frac{\partial(x_{n-1},\Gamma x_{n-1},a)\partial(x_{n-1},\Gamma x_n,a)}{1+\partial(x_{n-1},x_n,a)}$$

$$s\partial(x_n,x_{n+1},a)\le\alpha_1\partial(x_{n-1},x_n,a)+\alpha_2\partial(x_n,x_{n+1},a)+\alpha_5 s[\partial(x_{n-1},x_n,a)+\partial(x_n,x_{n+1},a)]$$

$$(s-\alpha_2-s\alpha_5)\partial(x_n,x_{n+1},a)\le(\alpha_1+s\alpha_5)\partial(x_{n-1},x_n,a) \tag{4}$$

Adding (3) and (4), we get

$$(2s-2\alpha_2-s\alpha_4-s\alpha_5)\partial(x_n,x_{n+1},a)\le(2\alpha_1+s\alpha_4+s\alpha_5)\partial(x_{n-1},x_n,a)$$

$$\partial(x_n,x_{n+1},a)\le\frac{(2\alpha_1+s\alpha_4+s\alpha_5)}{(2s-2\alpha_2-s\alpha_4-s\alpha_5)}\partial(x_{n-1},x_n,a)$$

$$\alpha=\frac{(2\alpha_1+s\alpha_4+s\alpha_5)}{(2s-2\alpha_2-s\alpha_4-s\alpha_5)}$$

In view of $(\alpha_1+\alpha_2+\alpha_3+s\alpha_4+s\alpha_5)<1$, then $(0\le\alpha<1$. Thus by the Lemma (2), $\{x_n\}$ is a b_2-Cauchy sequence in χ. Since (χ,∂) is a b_2-complete, then there exists some point x^* such that $x_n\to x^*$ as $n\to\infty$, then by (2), we can easily see that

$$s\partial(\Gamma x,\Gamma y,a)\le\alpha_1\partial(x_n,x^*,a)+\alpha_2\frac{\partial(x_n,\Gamma x_n,a)\partial(x^*,\Gamma x^*,a)}{1+\partial(x,y,a)}+\alpha_3\frac{\partial(x_n,\Gamma x^*,a)\partial(x^*,\Gamma x^*,a)}{1+\partial(x,y,a)}+$$
$$\alpha_4\frac{\partial(x_n,\Gamma x_n,a)\partial(x_n,\Gamma x^*,a)}{1+\partial(x,y,a)}+\alpha_5\frac{\partial(x^*,\Gamma x_n,a)\partial(x^*,\Gamma x^*,a)}{1+\partial(x,y,a)}$$

$$\begin{aligned}&=\alpha_1\partial(x_n,x^*,a)+\alpha_2\frac{\partial(x_n,x_{n+1},a)\partial(x^*,Tx^*,a)}{1+\partial(x_n,x^*,a)}+\alpha_3\frac{\partial(x_n,Tx^*,a)\partial(x^*,Tx^*,a)}{1+\partial(x_n,x^*,a)}+\\&\alpha_4\frac{\partial(x_n,x_{n+1},a)\partial(x^*,Tx^*,a)}{1+\partial(x_n,x^*,a)}+\alpha_5\frac{\partial(x^*,x_{n+1},a)\partial(x^*,Tx^*,a)}{1+\partial(x_n,x^*,a)}\end{aligned} \tag{5}$$

Taking the limit as $n \to \infty$ both the sides of (5), we get $\underset{n\to\infty}{Limit}(x_{n+1},\Gamma x^*) = 0$ i e. $x_n \to \Gamma x^* as, n \to \infty$.

It demonstrates that $\Gamma x^* = x^*$ by virtue of the limit of the b_2-convergent sequence's uniqueness.

x* is a fixed point of Γ as a result.

To demonstrate the fixed point's uniqueness, we assume that if there is a second fixed point y*, then by (2), we get

$$s\partial(\Gamma x^*,\Gamma y^*,a) \le \alpha_1\partial(x^*,y^*,a) + \alpha_2\frac{\partial(x^*,\Gamma x^*,a)\partial(y^*,\Gamma y^*,a)}{1+\partial(x^*,y^*,a)} + \alpha_3\frac{\partial(x^*,\Gamma y^*,a)\partial(y^*,\Gamma x^*,a)}{1+\partial(x^*,y^*,a)} + \alpha_4\frac{\partial(x^*,\Gamma x^*,a)\partial(x^*,\Gamma y^*,a)}{1+\partial(x^*,y^*,a)} + \alpha_5\frac{\partial(y,\Gamma x,a)\partial(y,\Gamma y,a)}{1+\partial(x^*,y^*,a)}$$

$$s\partial(\Gamma x^*,\Gamma y^*,a) \le \alpha_1\partial(x^*,y^*,a) + \alpha_3\partial(x^*,\Gamma y^*,a)$$

$$\partial(x^*,y^*,a) \le \frac{(\alpha_1+\alpha_3)}{s}(x^*,y^*,a) \tag{6}$$

As $(\alpha_1+\alpha_2+\alpha_3+s\alpha_4+s\alpha_5) < 1$, this implies that $(\alpha_1+\alpha_3) < 1, as, s \ge 1$.

We conclude from (6) that $\partial(x^*,y^*,a) = 0 \Rightarrow x^* = y^*$.

We now generalize Naidu's [6] finding.

Theorem 4: Let $(\chi,\le)$ be a partially ordered set and suppose that there exist a b_2- metric ∂ on χ such that (χ,∂) is a b_2- complete b_2- metric space with coefficient s≥1 and $\Gamma_1 \& \Gamma_2$ be a pair of self mapping from χ to χ satisfying the following conditions:

$$(a)\quad s[\partial^2(\Gamma_1x,\Gamma_2y,a)] \le \alpha\partial(x,\Gamma_1x,a)\partial(y,\Gamma_2y,a) + \beta\partial(y,\Gamma_1x,a)\partial(x,\Gamma_2y,a) - \min\{\partial(x,y,a)\partial(y,\Gamma_2y,a), \partial(x,\Gamma_1x,a)\partial(y,\Gamma_2y,a), \partial(x,\Gamma_1x,a)\partial(y,\Gamma_1x,a), \partial(x,\Gamma_2y,a)\partial(y,\Gamma_1x,a), \partial(y,\Gamma_1x,a)\partial(y,\Gamma_2y,a)\} \tag{7}$$

(b) $\Gamma_1 \& \Gamma_2$ are compatible pair for every x, y, a $\in \chi$ and for some non-negative $\begin{matrix}\alpha,\beta, with\\ 0\le\alpha,\beta \prec 1\end{matrix}$ and $\frac{\alpha+2\beta s}{s} = \lambda < 1$. Then $\Gamma_1 \& \Gamma_2$ have a common fixed point in χ. Further, if $\frac{\beta}{s} \prec 1$, then $\Gamma_1 \& \Gamma_2$ have a unique fixed point.

Proof: Let $\frac{\alpha+2\beta s}{s} = \lambda$, we define a sequence $\{x_n\}$ subset of χ for an arbitrary point $x_0 \in \chi$ such that $\Gamma_1x_n = x_{n+1}$, $\Gamma_2x_{n+1} = x_{n+2}$, n=0, 1, 2, 3,……

$$s[\delta^2(x_n,x_{n+1},a)] = s[\delta^2(\Gamma_1x_{n-1},\Gamma_2x_n,a)]$$

$$\le \alpha[\delta(x_{n-1},\Gamma_1x_{n-1},a)\delta(x_n,\Gamma_2x_n,a)] + \beta[\delta(x_n,\Gamma_1x_n,a)\delta(x_{n-1},\Gamma_2x_n,a)]$$

$$\text{-min}\left\{\begin{matrix}\delta(x_{n-1},x_n,a)\delta(x_n,\Gamma_2x_n,a), \delta(x_{n-1},\Gamma_1x_{n-1},a)\delta(x_n,\Gamma_2x_n,a),\\ \delta(x_{n-1},\Gamma_1x_{n-1},a)\delta(x_n,\Gamma_1x_{n-1},a), \delta(x_{n-1},\Gamma_2x_n,a)\delta(x_n,\Gamma_1x_{n-1},a),\\ \delta(x_n,\Gamma_2x_n,a)\delta(x_n,\Gamma_1x_{n-1},a)\}\end{matrix}\right.$$

$$s[\delta^2(x_n,x_{n+1},a)] \le \alpha[\delta(x_{n-1},x_n,a))\delta(x_n,x_{n+1},a)] + \beta[\delta(x_n,x_{n+1},a)s\{\delta(x_{n-1},x_n,a)+\delta(x_n,x_{n+1},a)\}]$$

$$\text{-min}\left\{\begin{matrix}\delta(x_{n-1},x_n,a)\delta(x_n,x_{n+1},a), \delta(x_{n-1},x_n,a)\delta(x_n,x_{n+1},a),\\ \delta(x_{n-1},x_n,a)\delta(x_n,x_n,a), \delta(x_{n-1},x_{n+1},a)\delta(x_n,x_n,a),\\ \delta(x_n,x_{n+1},a)\delta(x_n,x_n,a)\}\end{matrix}\right.$$

$$s[\delta^2(x_n,x_{n+1},a)] \le \alpha[\delta(x_{n-1},x_n,a))\delta(x_n,x_{n+1},a)] + \beta s[\delta(x_n,x_{n+1},a)\delta(x_{n-1},x_n,a)] + \beta s[\delta^2(x_n,x_{n+1},a)]$$

$$\text{-min}\left\{\begin{matrix}\delta(x_{n-1},x_n,a)\delta(x_n,x_{n+1},a), \delta(x_{n-1},x_n,a)\delta(x_n,x_{n+1},a),\\ 0,0,0\}\end{matrix}\right.$$

$$s(1-\beta)\delta(x_n, x_{n+1}, a) \le (\alpha + s\beta)\delta(x_{n-1}, x_n, a)$$

$$\delta(x_n, x_{n+1}, a) \le \frac{(\alpha + s\beta)}{s(1-\beta)}\delta(x_{n-1}, x_n, a)$$

$$(\alpha + s\beta) \prec s(1-\beta) \Rightarrow \lambda = \frac{\alpha + s\beta}{s} \prec 1$$

$$\delta(x_n, x_{n+1}, a) \le \lambda\delta(x_{n-1}, x_n, a) \tag{8}$$

Replacing x, y, a by x_{n-1}, x_n, a respectively, we have

$$\delta(x_{n-1}, x_n, a) \le \lambda\delta(x_{n-2}, x_{n-1}, a) \tag{9}$$

From (8) and (9), we have

$$\delta(x_n, x_{n+1}, a) \le \lambda^2\delta(x_{n-1}, x_n, a) \tag{10}$$

Continue in same manner, n times, we have

$$\delta(x_n, x_{n+1}, a) \le \lambda^n\delta(x_0, x_1, a)$$

Thus, for some $m, n \succ 0, m \succ n,$ we have

$$\delta(x_n, x_m, a) \le \delta(x_n, x_{n+1}, a) + \delta(x_{n+1}, x_{n+2}, a) + \delta(x_{n+2}, x_{n+3}, a) + ... + \delta(x_{m-1}, x_m, a)$$

$$\le (\lambda^n + \lambda^{n+1} + ... + \lambda^{m-1})\delta(x_0, x_1, a)$$

$$\le \frac{\lambda^n(1-\lambda^{m-n})}{1-\lambda}\delta(x_0, x_1, a).$$

Letting $m, n \to \infty,$ we have $\delta(x_n, x_m, a) \to 0.$

$\Rightarrow \{x_n\}$ is a Cauchy b_2- sequence in χ.

Again Γ_1, Γ_2 are compatible pair and $\{x_n\} \subseteq \chi$ is a sequence then $\{\delta(\Gamma_1\Gamma_2 x_n, \Gamma_2\Gamma_1 x_n)\} \to 0$ as $\{\Gamma_1 x_n\}$ and $\{\Gamma_2 x_n\}$ converges to same limit. So,

$\lim_{n\to\infty}\Gamma_1\Gamma_2 x_n = \lim_{n\to\infty}\Gamma_2\Gamma_1 x_n \Rightarrow \Gamma_1(\lim_{n\to\infty}\Gamma_2 x_n) = \Gamma_2(\lim_{n\to\infty}\Gamma_1 x_n) \Rightarrow \Gamma_1 u = \Gamma_2 u = u$, for some u i e. u is a common fixed point for $\Gamma_1 and \Gamma_2$

To demonstrate the fixed point's exclusivity, we assume that if there is a second fixed point v, then by (7), we obtain

$$s[\partial^2(\Gamma_1 u, \Gamma_2 v, a)] \le \alpha\partial(u, \Gamma_1 u, a)\partial(v, \Gamma_2 v, a) + \beta\partial(v, \Gamma_1 u, a)\partial(u, \Gamma_2 v, a) - \min\{\partial(u, v, a)\partial(v, \Gamma_2 v, a),$$
$$\partial(u, \Gamma_1 u, a)\partial(v, \Gamma_2 v, a), \partial(u, \Gamma_1 u, a)\partial(v, \Gamma_1 u, a), \partial(u, \Gamma_2 v, a)\partial(v, \Gamma_1 u, a), \partial(v, \Gamma_1 u, a)\partial(v, \Gamma_2 v, a)\}$$

$$\partial(u, v, a) \le \frac{\beta}{s}\partial(u, v, a)$$

$\Rightarrow \partial(u, v, a) = 0$ as $\frac{\beta}{s} \prec 1.$

$\Rightarrow u = v$. This completes the proof.

3. Conclusion

Zead Mustafa et al. [27] introduce the notion of b_2-metric spaces as a generalization of both 2-metric and b-metric spaces. He then proved a few fixed point theorems in a variety of contractive settings in partially ordered b_2-metric spaces.

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