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Separation Axioms in Soft Bitopological Ordered Spaces

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Abstract: This paper presents a comprehensive study on bi-ordered soft separation axioms applied to soft bitopological ordered spaces. The main focus of this research is to examine the properties, descriptions, and characteristics of these axioms. By exploring the relationships between these axioms and other properties of soft bitopological ordered spaces, this study expands our understanding of these spaces and their associated properties. Notably, significant findings are presented, establishing connections between the introduced bi-ordered axioms and properties such as soft bitopological and soft hereditary properties. The concepts of bi-ordered soft separation axioms, namely PST_i (resp. $PST_i^{\bullet}, PST_i^{*}, PST_i^{**}$)$-$ordered spaces, (where $i = 0, 1, 2$), are introduced and illustrated through relevant examples. These examples help clarify the relationships among the axioms and enhance our comprehension of their significance. Furthermore, this paper investigates the distinctions among separation axioms in topological ordered spaces and provides examples of relevant attributes from the literature. The separation axioms discussed in this research demonstrate enhanced descriptive power in characterizing the properties of topological ordered spaces. In addition to the above, the paper introduces the concept of bi-ordered subspace and explores the property of hereditary in the context of soft bitopological ordered spaces. These additions further enrich the understanding and applicability of bi-ordered soft separation axioms.

Keywords: Soft Set, Soft Singleton, $Bi-$ordered Soft Separation Axioms, $Bi-$ordered Subspace, Hereditary Property

1. Introduction

In 1965, Nachbin [17] introduced the concept of a topological ordered space by incorporating a partial order relation into the structure of a topological space, thereby generalizing the notion of a topological space. McCartan [13] later employed the concept of monotone neighborhoods to define and study ordered separation axioms in these spaces.

Real-life problems often involve vagueness and uncertainty, which has prompted the development of various mathematical tools to address these issues. Among these tools are fuzzy sets, intuitionistic fuzzy sets, rough sets, and vague sets. Another mathematical instrument designed to handle vagueness and uncertainty is soft sets, which was first introduced by Molodtsov [16] in 1999. Since its inception, soft set theory has been further developed and applied in decision-making problems by researchers such as Maji et al. in [14, 15]. In 2007, Aktas and Cagman [1] extended the application of soft set theory to algebraic structures.

In subsequent studies, the concept of soft separation axioms for crisp points was investigated by Shabir and Naz [19], while Hussain and Ahmad [7] examined properties related to soft interior, soft closure, and soft boundary. Nazmul and Samanta [18] studied neighborhood properties of soft topological spaces. Four different types of separation axioms in the context of soft topology were defined and discussed in a series of papers [8, 10, 19, 21], and Singh and Noorie [20] established connections between these types of spaces (T_i, $i = 1, 2, 3, 4$), further expanding the understanding of soft topological spaces and their properties.

In 2014, Ittanagi introduced the concept of soft bitopological spaces [9], which are defined over an initial universal set Υ with a fixed set of parameters Π. Ittanagi also introduced various types of soft separation axioms in this context. Kandil et al. [11] further studied the structures of soft bitopological spaces, defining fundamental concepts such as pairwise open (closed) soft sets and pairwise soft closure (interior, kernel) operators. They showed that the family of all pairwise open soft sets forms a supra soft topology η_{12} that includes η_1 and η_2, but it is not always a soft topology.

El-Shafei et al. [4, 5] introduced two new types of soft relations, "partial belong" and "total non-belong," and employed them to develop the concept of a "soft topological ordered space." They also presented the notion of "ordered soft separation axioms," specifically P-soft T_i-ordered spaces, where $i = 0, 1, 2, 3, 4$.

Additionally, El-Sheikh et al. [6] introduced the concept of soft bitopological ordered spaces, which includes increasing (decreasing, balancing) pairwise open (closed) soft sets, as well as the notions of increasing (decreasing, balancing) total (partial) pairwise soft neighborhoods and increasing (decreasing) pairwise open soft neighborhoods. They also studied the relationships between these concepts, including the increasing (decreasing) pairwise soft closure (interior).

In this paper, we explore the use of soft sets and soft topologies in the context of ordered spaces. We begin by providing definitions and properties of soft sets and soft topologies in Section 2 as a preliminary step. In Section 3, we introduce the concept of "bi-ordered soft separation axioms" called PST_i (resp. $PST_i^{\bullet}, PST_i^{*}, PST_i^{**}$)$-$ordered spaces, $(i = 0, 1, 2)$. We provide examples to illustrate the connections between these concepts and highlight their characteristics.

2. Preliminaries

To ensure clear understanding, specialized mathematical concepts such as "soft set, soft points, soft topological space, soft topological ordered space, and soft bitopological ordered space" will be explained concisely. Relevant references and resources for further reading include [4, 6, 12, 19]. Mathematical notation will be used, such as Υ to represent the set of all elements, Π to represent a specific set of values used to define the elements in Υ, and 2^{Υ} to denote the set of all subsets of Υ, for effective communication.

Definition 2.1. [12] A binary relation $\lesssim$ on Υ is considered a partial order relation if it satisfies the properties of reflexivity, anti-symmetry, and transitivity. The equality relation on Υ is represented by $\blacktriangle$ and consists of pairs of the form (ρ, ρ) for every ρ in Υ.

Definition 2.2. [17] A topological ordered space is defined as a triple $(\Upsilon, \eta, \lesssim)$, where (Υ, η) represents a topological space, and $(\Upsilon, \lesssim)$ represents a partially ordered set.

Definition 2.3. [16] A pair (ω, Π) constitutes a soft set over Υ when ω is a function mapping from Π to the power set of Υ, denoted as $\omega : \Pi \longrightarrow 2^{\Upsilon}$. For brevity, we employ the notation ω_{Π} instead of (ω, Π). Another representation of a soft set is as a collection of ordered pairs, $\omega_{\Pi} = \{(\alpha, \omega(\alpha)) : \alpha \in \Pi$ and $\omega(\alpha) \in 2^{\Upsilon}\}$. This implies that each element α in the set Π is mapped by the function ω to a subset of Υ, and ω_{Π} encompasses all such pairs $(\alpha, \omega(\alpha))$. The set comprising all soft sets over Υ is denoted as $P(\Upsilon)^{\Pi}$.

Definition 2.4. [15] Given $\omega_{\Pi} \in P(\Upsilon)^{\Pi}$, the following definitions hold:

1. A soft set ω_{Π} is referred to as a null soft set and denoted by $\widehat{\phi}$ if, for every α in Π, the function ω maps it to the empty set, i.e., $\omega(\alpha) = \emptyset$.
2. A soft set ω_{Π} is termed an absolute soft set and denoted by Υ_{Π}, $\omega(\alpha) = \Upsilon$, if, for each α in Π, the function ω maps it to the entire set Υ, i.e., $\omega(\alpha) = \Upsilon$.

Definition 2.5. [2] Let ω_{Π} and $\hbar_{\Pi}$ be soft sets in $P(\Upsilon)^{\Pi}$. The definitions are as follows:

1. $\hbar_{\Pi}$ is considered a soft subset of ω_{Π} and denoted by $\hbar_{\Pi} \sqsubseteq \omega_{\Pi}$ if, for every α in Π, the function $\hbar$ maps it to a subset of the set that ω maps it to.
2. The union of $\hbar_{\Pi}$ and ω_{Π} is a soft set λ_{Π}, denoted by $\hbar_{\Pi} \sqcup \omega_{\Pi}$, defined as $\lambda(\alpha) = \hbar(\alpha) \cup \omega(\alpha)$ for all α in Π.
3. The intersection of $\hbar_{\Pi}$ and ω_{Π} is a soft set λ_{Π}, denoted by $\hbar_{\Pi} \sqcap \omega_{\Pi}$, defined as $\lambda(\alpha) = \hbar(\alpha) \cap \omega(\alpha)$ for all α in Π.

Definition 2.6. [19] Let ω_{Π} and $\hbar_{\Pi}$ be soft sets in $P(\Upsilon)^{\Pi}$. The definitions are as follows:

1. The difference of $\hbar_{\Pi}$ and ω_{Π} is a soft set λ_{Π}, denoted by $\lambda_{\Pi} = \hbar_{\Pi} - \omega_{\Pi}$, defined as $\lambda(\alpha) = \hbar(\alpha) - \omega(\alpha)$ for all α in Π.
2. The complement of $\hbar_{\Pi}$, denoted by $\hbar_{\Pi}^{c}$, is defined as $\hbar^{c}(\alpha) = (\hbar(\alpha))^{c}$ for all α in Π.

Definition 2.7. [19] The soft set ν_{Π} over Υ is defined by a function ν, that maps each element α in the set Π to a set containing only the element ν, represented by $\nu(\alpha) = \nu$, for each $\alpha \in \Pi$.

Definition 2.8. [3] A soft set ω_{Π} over Υ is referred to as a soft singleton if there exists an element ν_0 in Υ such that $\omega(\alpha) = \nu_0$ for some α in Π. We denote a soft singleton as $\omega_{\Pi}^{\nu_0}$.

Definition 2.9. [4, 16] For a soft set $\hbar_{\Pi}$ over Υ and an element $\rho \in \Upsilon$,

1. We say $\rho \in \hbar_{\Pi}$ if $\rho \in \hbar(\alpha)$, for each $\alpha \in \Pi$ and $\rho \notin \hbar_{\Pi}$ if $\rho \notin \hbar(\alpha)$, for some $\alpha \in \Pi$.
2. We say $\rho \Subset \hbar_{\Pi}$ if $\rho \in \hbar(\alpha)$, for some $\alpha \in \Pi$ and $\rho \not\Subset \hbar_{\Pi}$ if $\rho \notin \hbar(\alpha)$, for each $\alpha \in \Pi$.

The symbols $\in, \notin, \Subset$, and $\not\Subset$ are interpreted as the relations of belonging, non-belonging, partial belonging, and total non-belonging, respectively.

Definition 2.10. [19] A soft topology on Υ is a collection η of soft sets over Υ with respect to Π that satisfies the following conditions:

1. The null soft set $\widehat{\phi}$ and the absolute soft set Υ_{Π} are elements of η.
2. The union of any soft sets in η is also in η.
3. The intersection of any two soft sets in η is also in η.

The triple (Υ, η, Π) is referred to as a soft topological space over Υ, where each element in η is called a soft open set and its relative complement is called a soft closed set.

Definition 2.11. [9] A soft bitopological space is defined as a quadruple $(\Upsilon, \eta_1, \eta_2, \Pi)$, where η_1 and η_2 are two distinct soft topologies defined on Υ, with a fixed set of parameters Π.

Definition 2.12. [11] In a soft bitopological space $(\Upsilon, \eta_1, \eta_2, \Pi)$, a soft set $\hbar_\Pi$ is called pairwise open (abbreviated as $PO-$soft) if it can be expressed as the union of a η_1-open soft set $\hbar^1_\Pi$ and a η_2-open soft set $\hbar^2_\Pi$. Similarly, a soft set $\hbar_\Pi$ is called pairwise closed (abbreviated as $PC-$soft) if its complement is a $PO-$soft set.

Definition 2.13. [4] A partially ordered soft space is defined as a triple $(\Upsilon, \Pi, \lesssim)$, where Υ is a set, Π is a set of parameters, and $\lesssim$ is a partial order relation on Υ.

Definition 2.14. [4] Let $(\Upsilon, \Pi, \lesssim)$ be a partially ordered soft space. An increasing soft operator $i : (P(\Upsilon)^\Pi, \lesssim) \to (P(\Upsilon)^\Pi, \lesssim)$ and a decreasing soft operator $d : (P(\Upsilon)^\Pi, \lesssim) \to (P(\Upsilon)^\Pi, \lesssim)$. For each soft set $\hbar_\Pi$ in $P(\Upsilon)^\Pi$: Let $(\Upsilon, \Pi, \lesssim)$ be a partially ordered soft space. An increasing soft operator i and a decreasing soft operator d are defined as mappings from $P(\Upsilon)^\Pi$ to $P(\Upsilon)^\Pi$. For a given soft set $\hbar_\Pi$ in $P(\Upsilon)^\Pi$, $i(\hbar_\Pi)$ is defined as $(i\hbar)_\Pi$, where $i\hbar$ maps elements of Π to subsets of Υ such that each element α is mapped to $\rho \in \Upsilon : \delta \lesssim \rho$, for some $\delta \in \hbar(\alpha)$. Similarly, $d(\hbar_\Pi)$ is defined as $(d\hbar)_\Pi$, where $d\hbar$ maps elements of Π to subsets of Υ such that each element α is mapped to $\rho \in \Upsilon : \rho \lesssim \delta$, for some $\delta \in \hbar(\alpha)$.

Definition 2.15. [4] In a partially ordered soft space $(\Upsilon, \Pi, \lesssim)$, a soft subset $\hbar_\Pi$ is said to be increasing if it satisfies $\hbar_\Pi = i(\hbar_\Pi)$, and it is called decreasing if $\hbar_\Pi = d(\hbar_\Pi)$.

Definition 2.16. [4] A quadrable system $(\Upsilon, \eta, \Pi, \lesssim)$ is referred to as a soft topological ordered space (STOS) if it satisfies two conditions: (Υ, η, Π) is a soft topological space, and $(\Upsilon, \Pi, \lesssim)$ is a partially ordered soft space.

Definition 2.17. [4] An increasing (resp. decreasing) soft neighborhood ε_Π of an element $\nu \in \Upsilon$ in an STOS $(\Upsilon, \eta, \Pi, \lesssim)$ is defined as a soft neighborhood of ν that is also an increasing (resp. decreasing) soft subset.

Definition 2.18. [4] Let $(\Upsilon, \eta, \Pi, \lesssim)$ be an STOS. We say it satisfies the following properties:

1. It is lower (resp. upper) P-soft T_1-ordered if for any distinct points $\nu, \zeta \in \Upsilon$, there exists an increasing (resp. decreasing) soft neighborhood ε_Π of ν such that $\zeta \notin \varepsilon_\Pi$.
2. It is P-soft T_0-ordered if it is either lower P-soft T_1-ordered or upper P-soft T_1-ordered.
3. It is P-soft T_1-ordered if it is both lower P-soft T_1-ordered and upper P-soft T_1-ordered.
4. It is P-soft T_2-ordered if for any distinct points $\nu, \zeta \in \Upsilon$, there exist disjoint soft neighborhoods ε_Π and V_Π of ν and ζ respectively, such that ε_Π is increasing and V_Π is decreasing.

Definition 2.19. [6] A soft bitopological ordered space (SBTOS) is defined as the system $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ satisfying the following conditions:

1. $(\Upsilon, \eta_1, \eta_2, \Pi)$ is a soft bitopological space.
2. $(\Upsilon, \Pi, \lesssim)$ is a partially ordered soft space.

Definition 2.20. [6] Let $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ be a $SBTOS$. A soft set M_Π over Υ is said to be:

1. Increasing pairwise open soft (briefly, $IPO-$soft) if $M_\Pi = M^1_\Pi \sqcup M^2_\Pi, M^\beta_\Pi \in \eta_\beta$ and increasing , $\beta = 1, 2$.
2. Decreasing pairwise open soft (briefly, $DPO-$soft) if $M_\Pi = M^1_\Pi \sqcup M^2_\Pi, M^\beta_\Pi \in \eta_\beta$ and decreasing , $\beta = 1, 2$.
3. Increasing pairwise closed soft (briefly, $IPC-$soft) if $M_\Pi = M^1_\Pi \sqcap M^2_\Pi, M^\beta_\Pi \in \eta^c_\beta$ and increasing , $\beta = 1, 2$
4. Decreasing pairwise closed soft (briefly, $DPO-$soft) if $M_\Pi = M^1_\Pi \sqcap M^2_\Pi, M^\beta_\Pi \in \eta^c_\beta$ and decreasing , $\beta = 1, 2$

Definition 2.21. [6] A soft set ε_Π in a $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is called:

1. Total pairwise soft neighborhood of $\rho \in \Upsilon$ if there is a $PO-$soft set M_Π such that $\rho \in M_\Pi \sqsubseteq \varepsilon_\Pi$.
2. Partial pairwise soft neighborhood of $\rho \in \Upsilon$ if there is a $PO-$soft set M_Π such that $\rho \Subset M_\Pi \sqsubseteq \varepsilon_\Pi$.

Definition 2.22. [6] A soft set ε_Π in a $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is called:

1. Increasing total pairwise soft neighborhood (briefly, $ITPS-$ nbd) of $\rho \in \Upsilon$ if ε_Π is a total pairwise soft neighborhood of $\rho \in \Upsilon$ and increasing.
2. Increasing partial pairwise soft neighborhood (briefly, $IPPS-$ nbd) of $\rho \in \Upsilon$ if ε_Π is a partial pairwise soft neighborhood of $\rho \in \Upsilon$ and increasing.
3. Decreasing total pairwise soft neighborhood (briefly, $DTPS-$ nbd) of $\rho \in \Upsilon$ if ε_Π is a total pairwise soft neighborhood of $\rho \in \Upsilon$ and decreasing.
4. Decreasing partial pairwise soft neighborhood (briefly, $DPPS-$ nbd) of $\rho \in \Upsilon$ if ε_Π is a partial pairwise soft neighborhood of $\rho \in \Upsilon$ and decreasing.

3. $Bi-$Ordered Soft Separation Axioms

This section introduces a novel concept known as Bi-ordered soft separation axioms or PST_i(resp. $PST^\bullet_i, PST^*_i, PST^{**}_i$)$-$ordered spaces (where i can be 0, 1, or 2). The primary objective of this section is to thoroughly investigate the key properties associated with this concept. To facilitate a better understanding, several examples will be provided to illustrate the relationships between these axioms and to demonstrate the outcomes derived from this investigation. Additionally, the concept of bi-ordered subspace will be introduced, and the property of hereditary in the context of soft bitopological ordered spaces will be explored.

Definition 3.1. An $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is said to be:

1. Lower pairwise soft T_1-ordered (briefly, $LPST_1-$ordered): For any distinct points ν and ζ in Υ such that $\nu \not\lesssim \zeta$ there exists an $ITPS-$ nbd ε_Π of ν such that $\zeta \notin \varepsilon_\Pi$.
2. Lower pairwise soft $T^\bullet_1-$ordered (briefly, $LPST^\bullet_1-$ordered): For any distinct points ν and ζ in Υ such that $\nu \not\lesssim \zeta$ there exists an $ITPS-$ nbd ε_Π of ν such that $y \not\Subset \varepsilon_\Pi$.
3. Lower pairwise soft T^*_1-ordered (briefly, $LPST^*_1-$ordered): For any distinct points ν and ζ in Υ such that $\nu \not\lesssim \zeta$ there exists an $IPPS-$ nbd ε_Π of ν such that $\zeta \not\Subset \varepsilon_\Pi$.

4. Lower pairwise soft $T_1^{**}-$ordered (briefly, $LPST_1^{**}-$ordered): For any distinct points ν and ζ in Υ such that $\nu \not\lesssim \zeta$ there exists an $IPPS-$ nbd ε_Π of ν such that $\zeta \notin \varepsilon_\Pi$.
5. Upper pairwise soft T_1-ordered (briefly, $UPST_1-$ordered): For any distinct points ν and ζ in Υ such that $\nu \not\lesssim \zeta$ there exists a $DTPS-$ nbd ε_Π of ζ such that $\nu \not\in \varepsilon_\Pi$.
6. Upper pairwise soft $T_1^{\bullet}-$ordered (briefly, $UPST_1^{\bullet}-$ordered): For any distinct points ν and ζ in Υ such that $\nu \not\lesssim \zeta$ there exists a $DTPS-$ nbd ε_Π of ζ such that $\nu \notin \varepsilon_\Pi$.
7. Upper pairwise soft $T_1^{*}-$ordered (briefly, $UPST_1^{*}-$ordered): For any distinct points ν and ζ in Υ such that $\nu \not\lesssim \zeta$ there exists a $DPPS-$ nbd ε_Π of ζ such that $\nu \not\in \varepsilon_\Pi$.
8. Upper pairwise soft $T_1^{**}-$ordered (briefly, $UPST_1^{**}-$ordered): For any distinct points ν and ζ in Υ such that $\nu \not\lesssim \zeta$ there exists a $DPPS-$ nbd ε_Π of ζ such that $\nu \notin \varepsilon_\Pi$.
9. PST_0-ordered space: An SBTOS is PST_0-ordered if it satisfies either $LPST_1-$ ordered or $UPST_1-$ordered.
10. $PST_0^{\bullet}-$ordered space: An SBTOS is $PST_0^{\bullet}-$ordered if it satisfies either $LPST_1^{\bullet}-$ ordered or $UPST_1^{\bullet}-$ordered.
11. $PST_0^{*}-$ordered space: An SBTOS is $PST_0^{*}-$ordered if it satisfies either $LPST_1^{*}-$ ordered or $UPST_1^{*}-$ordered.
12. $PST_0^{**}-$ordered space: An SBTOS is $PST_0^{**}-$ordered if it satisfies either $LPST_1^{**}-$ ordered or $UPST_1^{**}-$ordered.
13. PST_1-ordered space if it is $LPST_1-$ ordered and $UPST_1-$ ordered.
14. $PST_1^{\bullet}-$ordered space if it is $LPST_1^{\bullet}-$ ordered and $UPST_1^{\bullet}-$ ordered.
15. $PST_1^{*}-$ordered space: if it is $LPST_1^{*}-$ ordered and $UPST_1^{*}-$ ordered.
16. $PST_1^{**}-$ordered space if it is $LPST_1^{**}-$ ordered and $UPST_1^{**}-$ ordered.
17. PST_2-ordered space if for every distinct points ν, ζ in Υ such that $\nu \not\lesssim \zeta$ there exist disjoint total pairwise soft neighborhoods ε_Π and V_Π of ν and ζ, respectively, such that ε_Π is increasing and V_Π is decreasing.
18. $PST_2^{\bullet}-$ordered space if for every distinct points ν, ζ in Υ such that $\nu \not\lesssim \zeta$ there exist disjoint total pairwise soft neighborhood ε_Π of ν and partial pairwise soft neighborhood V_Π of ζ such that ε_Π is increasing and V_Π is decreasing.
19. $PST_2^{*}-$ordered space if for every distinct points ν, ζ in Υ such that $\nu \not\lesssim \zeta$ there exist disjoint partial pairwise soft neighborhoods ε_Π and V_Π of ν and ζ, respectively, such that ε_Π is increasing and V_Π is decreasing.
20. $PST_2^{**}-$ordered space if for every distinct points ν, ζ in Υ such that $\nu \not\lesssim \zeta$ there exist disjoint partial pairwise soft neighborhood ε_Π of ν and total pairwise soft neighborhood V_Π of ζ such that ε_Π is increasing and V_Π is decreasing.

Proposition 3.1. Every PST_1(resp. $PST_1^{\bullet}, PST_1^{*}, PST_1^{**}$) $-$ordered space $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is also a PST_0(resp. $PST_0^{\bullet}, PST_0^{*}, PST_0^{**}$)$-$ordered space.

Proof The proof is straightforward and follows directly from the definition 3.1

The following example is showing that the converse of the proposition is false by providing a specific counterexample.

Example 3.1. Let $\Pi = \{e_1, e_2\}, \lesssim = \blacktriangle \cup \{(\nu,\zeta),(\nu,z)\}$ be a partial order relation on $\Upsilon = \{\nu,\zeta,z\}$ and $\eta_1 = \{\widehat{\phi}, \Upsilon_\Pi, \omega_\Pi^1, \omega_\Pi^2, \omega_\Pi^3\}, \eta_2 = \{\widehat{\phi}, \Upsilon_\Pi, F_\Pi\}$ where,

$\omega_\Pi^1 = \{(e_1,\{\zeta\}),(e_2,\{\zeta\})\}$.
$\omega_\Pi^2 = \{(e_1,\{z\}),(e_2,\{z\})\}$.
$\omega_\Pi^3 = \{(e_1,\{\zeta,z\}),(e_2,\{\zeta,z\})\}$.
$F_\Pi = \{(e_1,\{\nu,\zeta\}),(e_2,\{\nu,\zeta\})\}$.

Then $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is $LPST_1$(resp. $LPST_1^{\bullet}, LPST_1^{*}, LPST_1^{**}$)$-$ ordered. So it is PST_0(resp. $PST_0^{\bullet}, PST_0^{*}, PST_0^{**}$)$-$ordered. On the other hand, every decreasing pairwise soft neighborhood of ν containing ζ. In simpler terms, this example is trying to prove that not all PST_0(resp. $PST_0^{\bullet}, PST_0^{*}, PST_0^{**}$)$-$ordered spaces are PST1-ordered spaces, by showing a specific example of a space that is $PST0-$ordered but not PST_1(resp. $PST_1^{\bullet}, PST_1^{*}, PST_1^{**}$)$-$ordered.

Proposition 3.2. Every PST_2(resp. PST_2^{**})$-$ordered space $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is also a $PST_1^{\bullet}$(resp. PST_1^{**})$-$ordered space.

Proof The proof directly follows from the definition 3.1.

The example that is being given is to show that the converse of this proposition is false.

Example 3.2. By taking $\eta_1 = \eta_2 = \eta$. The example is referring to an Example 4.7 in a previous work, [4]. It is stated that this example is PST_1-ordered (or $PST_1^{**}-$ordered) but not PST_2-ordered (or $PST_2^{**}-$ordered). This means that there exist PST_1-ordered (or $PST_1^{**}-$ordered) spaces that are not PST_2-ordered (or $PST_2^{**}-$ ordered), which contradicts the converse of the proposition.

Proposition 3.3. Every $PST_0^{\bullet}$(resp. $PST_1^{\bullet}, PST_0^{*}, PST_1^{*}$) $-$ordered space $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is also a PST_0(resp. $PST_1, PST_0^{**}, PST_1^{**}$)$-$ordered space.

Proof The proof relies on the observation that if a total non-belong relation $\notin$ exists, then it implies a non-belong relation $\not\in$.

The provided example serves to illustrate that the converse of this proposition is not true.

Example 3.3. Let $\Pi, \lesssim$ and Υ as in Example 3.1 and $\eta_1 = \{\widehat{\phi}, \Upsilon_\Pi, \omega_\Pi^1, \omega_\Pi^2, \omega_\Pi^3, \omega_\Pi^4\}, \eta_2 = \{\widehat{\phi}, \Upsilon_\Pi, F_\Pi^1, F_\Pi^2\}$ where,

$\omega_\Pi^1 = \{(e_1,\{\zeta\}),(e_2,\{\nu,\zeta\})\}$,
$\omega_\Pi^2 = \{(e_1,\{z\}),(e_2,\{\nu,z\})\}$,
$\omega_\Pi^3 = \{(e_1,\{\zeta,z\}),(e_2,\Upsilon)\}$,
$\omega_\Pi^4 = \{(e_1,\emptyset),(e_2,\{\nu\})\}$,
$F_\Pi^1 = \{(e_1,\{\nu\}),(e_2,\{\nu,\zeta\})\}$,
$F_\Pi^2 = \{(e_1,\emptyset),(e_2,\{\nu,\zeta\})\}$.

Now, $\eta_{12} = \eta_1 \cup \eta_2 \cup \{\lambda_\Pi^1, \lambda_\Pi^2, \lambda_\Pi^3\}$ where,

$\lambda_\Pi^1 = \{(e_1,\{\nu,\zeta\}),(e_2,\{\nu,\zeta\})\}$,
$\lambda_\Pi^2 = \{(e_1,\{\nu,z\}),(e_2,\Upsilon)\}$,

$\lambda^3_\Pi = \{(e_1, \{z\}), (e_2, \Upsilon)\}$.

In simple terms, this example is trying to prove that not all $PST_0^\bullet$ (resp. $PST_1^\bullet, PST_0^*, PST_1^*$)$-$ordered spaces are PST_0 (resp. $PST_1, PST_0^{**}, PST_1^{**}$)$-$ordered spaces, by showing a specific example of a space that is $PST_0^\bullet$ (resp. $PST_1^\bullet, PST_0^*, PST_1^*$)$-$ordered but not PST_0 (resp. $PST_1, PST_0^{**}, PST_1^{**}$)$-$ordered.

Proposition 3.4. Every PST_2-ordered space $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is PST_2^*-ordered.

Proof The proof for the proposition states that the belong relation $\in$ implies a total belong relation $\Subset$.

Example 3.4. Let $\Pi = \{e_\alpha, e_\beta\}$ be a set of parameters, $\lesssim = \blacktriangle \cup \{(1, 2)\}$ be a partial order relation on the set of natural numbers $\aleph$. Define $\eta_1 = \{\omega_\Pi \sqsubseteq \aleph_\Pi$ such that $1 \notin \omega_\Pi\}$ and $\eta_2 = \{F_\Pi \sqsubseteq \aleph_\Pi$ such that $2 \in \omega_\Pi\}$. The example states that this specific space is PST_2^*-ordered but not PST_2-ordered.

Proposition 3.5. Every PST_2 (resp. PST_2^{**})$-$ordered space $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is $PST_2^\bullet$ (resp. PST_2^*)$-$ordered.

Proof The proof for the proposition states that the belong relation $\in$ implies a total belong relation $\Subset$.

Example 3.5. The example provided states that it follows from an earlier example (Example 3.3) that a specific space is $PST_2^\bullet$ (resp. PST_2^*)$-$ordered but not PST_2 (resp. PST_2^{**})$-$ordered. However without the context of example 3.3 it is hard to understand the example provided.

Proposition 3.6. Every $PST_0^\bullet$ (resp. $PST_1^\bullet, PST_2^\bullet, PST_2,$ PST_2^*, PST_2^{**})$-$ordered space $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is also a PST_0^{**} (resp. $PST_1^{**}, PST_1^{**}, PST_1^*, PST_0^*, PST_0^{**}$)$-$ ordered space.

Proof It is based on the principle that belong relation $\in$ implies a total belong relation $\Subset$ and a total non belong relation $\not\Subset$ implies a non belong relation $\notin$.

Example 3.6. It follows from Example 3.3, illustrates that a specific space is PST_0^{**} (resp. $PST_1^{**}, PST_1^*, PST_0^*, PST_0^{**}$) $-$ ordered but not $PST_0^\bullet$ (resp. $PST_1^\bullet, PST_2^\bullet, PST_2, PST_2^*,$ PST_2^{**})$-$ordered.

The diagram illustrates the relationship between different types of separation axioms, as well as the implications between them as described in this paper.

$PST_1^\bullet \longrightarrow \quad PST_1 \longrightarrow \quad PST_0 \longrightarrow \quad PST_0^\bullet$

$\not\longleftarrow \qquad\qquad \not\longleftarrow \qquad \not\longleftarrow$

$\not\uparrow\downarrow \qquad \uparrow\not\downarrow \qquad \not\uparrow\not\downarrow \qquad \not\uparrow\downarrow$

$PST_1^{**} \not\longrightarrow \quad PST_1^* \longrightarrow \quad PST_0^* \longrightarrow \quad PST_0^{**}$

$\longleftarrow \qquad\qquad \not\longleftarrow \qquad \not\longleftarrow$

$\uparrow\not\downarrow \qquad \uparrow\not\downarrow \qquad \uparrow\not\downarrow \qquad \uparrow\not\downarrow$

$PST_2^\bullet \not\longrightarrow \quad PST_2 \longrightarrow \quad PST_2^* \not\longrightarrow \quad PST_2^{**}$

$\longleftarrow \qquad\qquad \not\longleftarrow \qquad\qquad \longleftarrow$

Theorem 3.1. Let $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ be an $SBTOS$. Then the following three statements are equivalent:

1. The space is $UPST_1^\bullet$(resp. $LPST_1^\bullet$)$-$ordered,
2. For any two elements ν and ζ in Υ such that $\nu \not\lesssim \zeta$, there is a pairwise soft open set ω_Π containing ζ (resp. ν) in which $\nu \not\lesssim z$(resp. $z \not\lesssim y$) for every $z \Subset \omega_\Pi$,
3. For any ν in Υ, the set $(i(\nu))_\Pi$ (resp. $d(\nu))_\Pi$) is pairwise soft closed.

Proof $(1 \to 2)$ If $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is an $UPST_1^\bullet-$ordered space, and ν and ζ are elements of Υ such that $\nu \not\lesssim \zeta$. Then there exists a $DTPS-$nbd ε_Π of ζ such that $\nu \not\Subset \varepsilon_\Pi$. Putting $\omega_\Pi = sint(\varepsilon_\Pi)$. Suppose that $\omega_\Pi \not\sqsubseteq (i(\nu))^c_\Pi$. Then there exists $z \Subset \omega_\Pi$ and $z \not\Subset (i(\nu))^c_\Pi$. It follows that $z \in (i(\nu))_\Pi$, which implies that $\nu \lesssim z$. Now, $z \Subset \omega_\Pi \sqsubseteq \varepsilon_\Pi$ implies that $\nu \Subset \varepsilon_\Pi$. However, this contradicts the fact that $\nu \not\Subset \varepsilon_\Pi$. Thus $\omega_\Pi \sqsubseteq (i(\nu))^c_\Pi$. Hence $\nu \not\lesssim z$, for every $z \Subset \omega_\Pi$.

$(2 \to 3)$ Consider $\nu \in \Upsilon$ and let $\rho \Subset (i(\nu))^c_\Pi$. Then $\nu \not\lesssim \rho$. Therefore there exists a $PO-$ soft set ω_Π containing ρ such that $\omega_\Pi \sqsubseteq (i(\nu))^c_\Pi$. Given that ν and ρ are picked without any specific criteria, then a pairwise soft set $(i(\nu))^c_\Pi$ is $PO-$ soft, for $\nu \in \Upsilon$. Hence $(i(\nu))_\Pi$ is $PC-$ soft, for any $\nu \in \Upsilon$.

$(3 \to 1)$ Let $\nu \not\lesssim \zeta \ \in X$ Obviously, $(i(\nu))_\Pi$ is increasing and by hypothesis, $(i(\nu))_\Pi$ is $PC-$ soft. Then $(i(\nu))^c_\Pi$ is a decreasing $PO-$soft soft set satisfies that $\zeta \in (i(\nu))^c_\Pi$ and $\nu \not\Subset (i(\nu))^c_\Pi$.

Thus, the proof is finished.

An analogous proof can be applied for the case inside the parentheses.

Proposition 3.7. If ν is the smallest (resp. the largest) element of a $LPST_1^\bullet$ (resp. $UPST_1^\bullet$)$-$ordered space $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$, then Υ_Π is decreasing (resp. increasing) $PC-$soft.

Proposition 3.8. If ν is the smallest (resp. the largest) element of a finite $PST_1^\bullet$ ordered space $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$, then Υ_Π is $DPO-$soft (resp. $IPO-$soft).

Proof The proposition is verified when ν is the smallest element, and the other case can be proved analogously. Since ν is the smallest element of X. Then $\nu \lesssim \zeta, \forall \zeta \in \Upsilon$. By the anti-symmetric of $\lesssim$, we have $\zeta \not\lesssim \nu, \forall \zeta \in \Upsilon$. By hypothesis, there is a $DTPS-$nbd F_Π of ν such that $\zeta \not\Subset F_\Pi$. It follows that $\Upsilon_\Pi = \sqcap F_\Pi$. Since Υ is finite, then Υ_Π is $DPO-$soft.

A parallel argument can be made for the situation inside the parentheses.

Proposition 3.9. If ν is the smallest (resp. the largest) element of a finite a PST_1^*-ordered space $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$, then F^ν_Π is $DPO-$soft (resp. $IPO-$soft).

Proof The proof is analogous to Proposition 3.8, with the substitution of ν_Π by F^ν_Π.

The aforementioned Proposition can be established in the scenario where $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is a finite PST_1^{**}-ordered space.

Proposition 3.10. A finite $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is $PST_1^\bullet-$ordered if and only if it is PST_2-ordered.

Proof Necessity: For each $\zeta \in (i(\nu))^c_\Pi$, we have $(d(\zeta))_\Pi$ is $PC-$ soft. Since Υ is finite, then $\sqcup_{\zeta \in (i(\nu))^c_\Pi} d(\zeta)$ is $PC-$ soft. Therefore $(\sqcup_{\zeta \in (i(\nu))^c_\Pi} d(\zeta))^c = (i(\nu))_\Pi$ is a $PO-$ soft set. Thus $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is a PST_2-ordered space.

Sufficiency: It directly follows from Proposition 3.2.

Proposition 3.11. Let $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ be an $SBTOS$ with $\eta_1 = \eta_2 = \eta$. If $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is $PST_i^\bullet-$ordered, then $(X, \eta, E, \lesssim)$ is always $P-$soft T_i-ordered, for $i = 0, 1$.

Proof We have shown the proposition when $i = 1$, and the other instance can be shown similarly. Let ν, ζ be two distinct points in $(\Upsilon, \eta, \Pi, \lesssim)$ such that $\nu \lesssim \zeta$. As $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is $PST_1^\bullet$, then there exist an $ITPS-$nbd ε_Π of ν such that $\zeta \not\Subset \varepsilon_\Pi$ and a $ITPS-$nbd F_Π of ζ such that $\nu \not\Subset F_\Pi$. Since $\eta_1 = \eta_2 = \eta$, then ε_Π is an increasing total soft neighborhood of x such that $\zeta \notin \varepsilon_{\Pi}$ and F_{Π} is a decreasing total soft neighborhood of ζ such that $\nu \notin F_{\Pi}$ in $(\Upsilon, \eta, \Pi, \lesssim)$. Thus $(\Upsilon, \eta, \Pi, \lesssim)$ is $P-$soft T_1-ordered.

Proposition 3.12. Let $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ be an $SBTOS$ with $\eta_1 = \eta_2 = \eta$. If $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is PST_2-ordered, then $(\Upsilon, \eta, \Pi, \lesssim)$ is always $P-$soft T_2-ordered.

Proof The proof is analogous to Proposition 3.11.

Definition 3.2. Let $\Omega \subseteq \Upsilon$ and $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ be an $SBTOS$. Then $(\Omega, \eta_{1\Omega}, \eta_{2\Omega}, \Pi, \lesssim_{\Omega})$ is called soft $bi-$ordered subspace of $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ provided that $(\Omega, \eta_{1\Omega}, \eta_{2\Omega}, \Pi)$ is soft bitopological subspace of $(\Upsilon, \eta_1, \eta_2, \Pi)$ and $\lesssim_{\Omega} = \lesssim \cap \Omega \times \Omega$.

Lemma 3.1. If U_{Π} is an increasing (resp. a decreasing) pairwise soft subset of an $SBTOS(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$, then $U_{\Pi} \sqcap \Omega_{\Pi}$ is an increasing (resp. a decreasing) pairwise soft subset of a soft $bi-$ordered subspace $(\Omega, \eta_{1\Omega}, \eta_{2\Omega}, \Pi, \lesssim_{\Omega})$.

Proof Let U_{Π} be an increasing pairwise soft subset of an $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$. In a soft $bi-$ordered subspace $(\Omega, \eta_{1\Omega}, \eta_{2\Omega}, \Pi, \lesssim_{\Omega})$, let $\rho \in i_{\lesssim_{\Omega}}(U_{\Pi} \sqcap \Omega_{\Pi})$. Since $i_{\lesssim_{\Omega}}(U_{\Pi} \sqcap \Omega_{\Pi}) \sqsubseteq i_{\lesssim_{\Omega}}(U_{\Pi}) \sqcap i_{\lesssim_{\Omega}}(\Omega_{\Pi}) \sqsubseteq U_{\Pi} \sqcap \Omega_{\Pi}$, then $\rho \in (U_{\Pi} \sqcap \Omega_{\Pi})$.Therefore$i_{\lesssim_{\Omega}}(U_{\Pi} \sqcap \Omega_{\Pi}) = U_{\Pi} \sqcap \Omega_{\Pi}$. Thus $U_{\Pi} \sqcap \Omega_{\Pi}$ is an increasing pairwise soft subset of a soft $bi-$ordered subspace $(\Omega, \eta_{1\Omega}, \eta_{2\Omega}, \Pi, \lesssim_{\Omega})$.

The demonstration is parallel in the case where U_{Π} is decreasing.

Theorem 3.2. The property of being a PST_i (resp. $PST_i^{\bullet}, PST_i^{*}, PST_i^{**})-$ordered space is hereditary, for $i = 0, 1, 2$.

Proof We establish the theorem for the case $i = 2$, and the other two scenarios can be demonstrated in a similar way. Let $(\Omega, \eta_{1\Omega}, \eta_{2\Omega}, \Pi, \lesssim_{\Omega})$ be a soft $bi-$ordered subspace of a PST_2 (resp. $PST_2^{\bullet}, PST_2^{*}, PST_2^{**})-$ordered space $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$. If $\rho, \delta \in \Omega$ such that $\rho \lesssim_{\Omega} \delta$, then $\rho \lesssim \delta$. So by hypothesis, there exist disjoint soft neighborhoods ε_{Π} and V_{Π} of ρ and δ, respectively, such that ε_{Π} is increasing and V_{Π} is decreasing. Setting $U_{\Pi} = \Omega_{\Pi} \sqcap \varepsilon_{\Pi}$ and $\omega_{\Pi} = \Omega_{\Pi} \sqcap V_{\Pi}$, by Lemma 3.1, we infer that U_{Π} is an increasing pairwise soft neighborhood of ρ and ω_{Π} is a decreasing pairwise soft neighborhood of δ. Since the soft neighborhoods U_{Π} and ω_{Π} are disjoint, it follows that $(\Omega, \eta_{1\Omega}, \eta_{2\Omega}, \Pi, \lesssim_{\Omega})$ is PST_2 (resp. $PST_2^{\bullet}, PST_2^{*}, PST_2^{**})-$ordered.

The theorem can be proved analogously when $i = 0, 1$.

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